ENDOMORPHISMS OF PROJECTIVE VARIETIES

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ABSTRACT. We study complex projective manifolds X that admit surjective endomorphisms $f : X \to X$ of degree at least two. In case f is étale, we prove structure theorems that describe X. In particular, a rather detailed description is given if X is a uniruled threefold. As to the ramified case, we first prove a general theorem stating that the vector bundle associated to a Galois covering of projective manifolds is ample (resp. nef) under very mild conditions. This is applied to the study of ramified endomorphisms of Fano manifolds with $b_2 = 1$. It is conjectured that $\mathbb{P}_n$ is the only Fano manifold admitting admitting an endomorphism of degree $d \geq 2$, and we prove that in several cases.

A part of the argumentation is based on a new characterization of $\mathbb{P}_n$ as the only manifold that admits an ample subsheaf in its tangent bundle.

CONTENTS

1. INTRODUCTION

A classical question in complex geometry asks for a description of projective manifolds X that admit surjective endomorphisms $f : X \to X$ of degree at least two —we refer to [Fak03] for questions and results relating endomorphisms of algebraic varieties with some general conjectures in number theory. A straightforward argument proves that X cannot be

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of general type and that f is necessarily finite, see [Bea01, Prop. 2] and [Fuj02, Lem. 2.3]. While a complete classification of all possible X remains an extremely hard open problem, the present paper presents a number of results. Two different approaches are highlighted in the text, according to whether the given endomorphism is étale or not.

1.A. **Étale endomorphisms and the minimal model program.** The first approach, started in Section 3, is based on the Minimal Model Program and deals mainly with étale endomorphisms. This approach was also used by Fujimoto in [Fuj02]. One can check without much difficulty that the direct image map $f_* : H_2(X, \mathbb{R}) \to H_2(X, \mathbb{R})$ permutes the geometric extremal rays of the Mori cone, and the same is true for extremal rays if f is étale, cf. Proposition 3.2 and [Fuj02, Prop. 4.2]. The targets of the associated contractions are related by finite morphisms, cf. Corollaries 3.4–3.6, Proposition 3.7 and [Fuj02, Prop. 4.4]. In Section 4, we prove any endomorphism of X is be étale if K_X is pseudo-effective. This generalizes a previous result by Iitaka [Iit82, Thm. 11.7]; see also [Bea01]. In Section 5, we employ the results of Sections 3 and 4 to study étale endomorphisms in greater detail. We observe that the existence of an étale endomorphism f has some implications for invariants of X. For example, the top self-intersection of the canonical bundle, the Euler characteristic, and the top Chern class of the manifold must all vanish, cf. Lemma 5.1. As a general result, we prove in Proposition 5.2 that all deformations of étale endomorphisms come from automorphisms of X. Furthermore, we study the Minimal Model Program in detail if X has dimension three. Since the case on non-negative Kodaira dimension was carefully treated in [Fuj02], we focus on the case of negative Kodaira dimension.

1.B. **The vector bundle associated with an endomorphism.** The second approach is inspired by Lazarsfeld's work [Laz80], see also [Laz04, 6.3.D]. The idea is to study a ramified finite covering f of degree d through the properties of the canonically associated vector bundle $\mathcal{E}_f$. Notably, the bundle $\mathcal{E}_f$ tends to inherit positivity properties from the ramification divisor. This works particularly well for Galois covers, as shown in the following Theorem.

Theorem 1.1 (cf. Theorem 6.7)**.** *Let $f : X \to Y$ be a Galois covering of projective manifolds which does not factor through an étale covering. Assume that all irreducible components of the ramification divisor are ample. Then $\mathcal{E}_f$ is ample.* □

The condition that f does not factor through an étale covering is automatically satisfied, e.g., for Fano manifolds X with $b_2(X) = 1$. Notice that the theorem is false without the Galois assumption.

The approach via the bundle $\mathcal{E}_f$ is applied in the last sections of the paper, where we investigate manifolds of negative Kodaira dimension admitting a ramified endomorphism. It is generally believed that the projective space is the only Fano manifold with Picard number one for which such non-trivial endomorphisms exist.

Conjecture 1.2. *Let $f : X \to X$ be an endomorphism of a Fano manifold X with $\rho(X) = 1$. If $\deg f > 1$, then $X \simeq \mathbb{P}_n$.*

At present, Conjecture 1.2 known to be true in the following special cases: surfaces, threefolds [ARVdV99, Sch99, HM03], rational homogeneous manifolds [PS89, HM99], or toric varieties [OW02], varieties containing a rational curve with trivial normal bundle [HM03, Cor. 3]. In Section 7, we enlarge the list. In particular, we prove the following results.

Theorem 1.3 (Indirect evidence for Conjecture 1.2)**.** *Let X be a Fano manifold with $\rho(X) = 1$ and $f : X \to X$ an endomorphism. If one of the following conditions holds, then* $\deg f = 1$.

- *X has index ≤ 2 and additionally there exists a line in X which is not contained in the branch locus of f —see Theorem 7.6*
- *X satisfies the Cartan-Fubini condition, X is almost homogeneous and $h^0(X, T_X) > \dim X$ —see Theorem 7.19*
- *X satisfies the Cartan-Fubini condition, X is almost homogeneous and either branch or the ramification divisor of f meets the open orbit of $\mathrm{Aut}^0(X)$ —see Theorem 7.19*
- *X is a del Pezzo manifold of degree 5 and $\rho(X) = 1$ —see Theorem 7.21.* □

Theorem 1.4 (Direct evidence for Conjecture 1.2)**.** *Let X be a Fano manifold with $\rho(X) = 1$ and $f : X \to X$ an endomorphism of degree $\deg f \geq 2$. Then $X \simeq \mathbb{P}_n$ if one of the following conditions hold.*

- *$\mathcal{E}_{f_k}$ is ample for some iterate f_k of f and $h^0(X,\, f_k^*(T_X)) > h^0(X,\, T_X)$ —see Corollary 7.9*
- *f is Galois and $h^0(X,\, f_k^*(T_X)) > h^0(X,\, T_X)$ for some k —see Corollary 7.11*
- *X is almost homogeneous and $h^0(X,\, T_X) > \dim X$ and $\mathcal{E}_{f_k}$ is ample for sufficiently large k —see Corollary 7.9 and Remark 7.16.1.* □

In order to check that X is isomorphic to a projective space, we prove the following partial generalization of a theorem of Andreatta and Wisniewski, [AW01].

Theorem 1.5 (cf. Theorem A.2)**.** *Let X be a projective manifold with $\rho(X) = 1$. Let $\mathcal{F} \subset T_X$ be a coherent subsheaf of positive rank. If $\mathcal{F}$ is ample, then $X \simeq \mathbb{P}_n$.* □

In fact, we prove a much stronger theorem, assuming the ampleness of $\mathcal{F}$ only on certain rational curves.

2. Notation and general facts

We collect some general facts on surjective endomorphisms. Unless otherwise noted, we fix the following assumptions and notation throughout the present work.

Assumption / Notation 2.1. Let X be a projective manifold and $f : X \to X$ a surjective endomorphism. Let d be the degree of f. The ramification divisor upstairs is denoted by $\mathfrak{R}$, so that

$$K_X = f^*(K_X) + \mathfrak{R}. \tag{2.1.1}$$

The branch divisor downstairs is called $\mathfrak{B}$. It is defined as the cycle-theoretic image $\mathfrak{B} = f_*(\mathfrak{R})$.

We briefly recall two lemmas that hold on every compact manifold and do not require any projectivity assumption.

Lemma 2.2 ([Bea01, Lem. 1], [Fuj02, Lem. 2.3])**.** *The linear maps*

$$f^* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \quad \textit{and} \quad f_* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

are isomorphisms. More precisely, $f_* f^* = d \cdot \mathrm{id}$, $f^* f_* = d \cdot \mathrm{id}$, and therefore $(f^*)^{-1} = \frac{1}{d} f_*$. In particular, f is finite. □

Lemma 2.3 ([Bea01, Prop. 2])**.** *Under the Assumption 2.1, the manifold X is not of general type, i.e., $\kappa(X) < \dim X$.* □

Étale endomorphisms and the minimal model program

3. EXTREMAL CONTRACTIONS OF MANIFOLDS WITH ENDOMORPHISMS

3.A. **Extremal contractions.** Recall that $N^1(X) \subset H^2(X, \mathbb{R})$ is the subspace generated by the classes of irreducible hypersurfaces and that $\overline{NE}(X) \subset N_1(X, \mathbb{R}) \subset H_2(X, \mathbb{R})$ is the closed cone generated by classes of irreducible curves. Equivalently,

$$\overline{NE}(X) = \big\{\alpha \in N_1(X, \mathbb{R}) \,|\, H.\alpha \geq 0 \text{ for all ample } H \in \operatorname{Pic}(X)\big\}.$$

Notation 3.1. A half-ray $R \subset \overline{NE}(X)$ is called *extremal* if it is geometrically extremal and if $K_X \cdot R < 0$. We say that $\alpha \in \overline{NE}(X)$ is extremal if the ray $R = \mathbb{R}_+\alpha$ is extremal.

We recall a proposition that has already been shown by Fujimoto. For the convenience of the reader, a short argument is included.

Proposition 3.2 (cf. [Fuj02, Prop. 4.2])**.** *Under the Assumption 2.1, the linear isomorphism of* $\mathbb{R}$*-vector spaces,* $f_* : H_2(X, \mathbb{R}) \to H_2(X, \mathbb{R})$*, restricts to a bijective map*

$$f_* : \overline{NE}(X) \to \overline{NE}(X).$$

In particular, f_* *defines a bijection on the set of geometrically extremal rays.*

(3.2.1) *If* $R \subset \overline{NE}(X)$ *is an extremal ray such that the exceptional locus* E_R *of the associated contraction is not contained in the ramification locus* $\mathfrak{R}$*, then* $f_*(R)$ *is again extremal.*

(3.2.2) *If* f *is étale, then* f_* *defines a bijection on the set of extremal rays.*

Proof. To show that f_* defines a bijection on the set of geometrically extremal rays, it suffices to note that both the cycle-theoretic image and preimage of an irreducible, effective curve under the finite morphism f is effective.

For Statement (3.2.1), let $R \subset \overline{NE}(X)$ be an extremal ray with associated exceptional set $E_R \subset X$. If $E_R \not\subset \mathfrak{R}$, let $C \subset E_R$ be a curve with $[C] \in R$ and $C \not\subset \mathfrak{R}$. Then

$$f_*([C]).K_X = [C].f^*(K_X) = \underbrace{K_X \cdot C}_{<0} - \underbrace{\mathfrak{R} \cdot C}_{\geq 0} < 0.$$

It follows that $f_*(R) = \mathbb{R}^+ \cdot f_*\big([C]\big)$ is extremal. This shows the second statement.

For (3.2.2), assume that f is étale. We will only need to show that the pull-back of an extremal curve is extremal. To that end, let R be an extremal ray. It is them immediately clear that $f^*(R)$ is geometrically extremal. To show that it is extremal, let $C \subset X$ be an irreducible rational curve with $[C] \in R$. Since

$$K_X \cdot f^*([C]) = f^*(K_X) \cdot f^*([C]) = d \cdot (K_X \cdot [C]) < 0,$$

the pull-back ray $f^*(R) = \mathbb{R}^+ \cdot f^*\big([C]\big)$ is thus indeed extremal. □

Remark 3.3. Under the Assumptions 2.1 suppose that X has only finitely many extremal rays. Then there exists a number k, with the following property: if f_k is the k^{th} iteration of f and if R is any extremal ray, then $(f_k)_*(R) = R$. Namely, if M is the finite set of extremal rays, then $(f_*)^k = (f_k)_* : M \to M$ is bijective. Hence there exists a number k such that f_*^k is the identity.

Corollary 3.4. *Let* $R \subset \overline{NE}(X)$ *be an extremal ray. If the associated contraction is birational, assume that the exceptional set* E_R *of the associated contraction is not contained*

in $\mathfrak{R}$. Then there exists a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[r, "{\phi,\ \text{contraction of } R}"] \arrow[d, "{f,\ \text{finite}}"'] & Y \arrow[d, "{g,\ \text{finite}}"] \\ X \arrow[r, "{\psi,\ \text{contraction of } f_*(R)}"'] & Z \end{tikzcd}$$

In particular, if $E_{f_*(R)}$ is the exceptional set of ψ, then $f(E_R) = E_{f_*(R)}$ and $E_R = f^{-1}\big(E_{f_*(R)}\big)$.

Remark 3.4.1. The ϕ-exceptional set E_R is defined as the set where ϕ is not locally isomorphic. If ϕ is of fiber type, then $E_R = X$.

Proof. Observe that if ϕ contracts a curve $C \subset X$, then ψ contracts $f(C) \subset X$, because the class $[f(C)]$ is contained in $f_*(R)$. Using Zariski's main theorem, this already shows the existence of g and proves that $f(E_R) \subset E_{f_*(R)}$. Since $b_2(Y) = b_2(Z) = b_2(X) - 1$, g is necessarily finite.

To show that $f(E_R) \supset E_{f_*(R)}$ and $E_R = f^{-1}\big(E_{f_*(R)}\big)$, let $C' \subset X$ be a curve which is contracted by ψ. Its class $[C']$ is then contained in $f_*(R)$, and $f^{-1}(C')$ is a union of curves whose individual classes are, by geometric extremality, contained in R. In particular, all components of $f^{-1}(C')$ are contained in E_R. $\square$

Corollary 3.5. *In the setup of Corollary 3.4, assume that $f : X \to X$ is étale. If $\dim X \geq 5$, assume additionally that the contraction ϕ is divisorial. Then the restriction $f|_{E_R} : E_R \to E_{f_*(R)}$ is étale of degree d. In particular, $E_{f_*(R)}$ is not simply connected.*

Proof. If ϕ is of fiber type, there is nothing to show. We will thus assume that ϕ is birational. Observe that if the contraction ϕ is divisorial, the statement follows from Corollary 3.4 and from the fact that the exceptional divisor of a divisorial contraction is irreducible. We consider the possibilities for $\dim X$.

If $\dim X \leq 3$, it follows from the classification of extremal contractions in dimension 3, [Mor82, Thm. 3.3, Thm. 3.5], that ϕ is divisorial. The claim is thus shown.

If $\dim X = 4$, we are again finished if that show that E_R is a divisor. We assume to the contrary and suppose $\dim E_R < 3$. In this setup, a theorem of Kawamata [Kaw89, Thm. 1.1], asserts that the exceptional loci of both E_R and $E_{f_*(R)}$ are disjoint copies of $\mathbb{P}_2$'s. In particular, they are simply connected. So, if $E_{f_*(R)}$ has n connected components, then $E_R = f^{-1}\big(E_{f_*(R)}\big)$ will have $d \cdot n$ connected components. But the same argumentation applies to the étale morphism $f \circ f : X \to X$ and yields that E_R has $d^2 \cdot n$ connected components. Again, we found a contradiction.

Finally, assume that $\dim X \geq 5$. Then ϕ is divisorial by assumption. $\square$

Corollary 3.6. *In the setup of Corollary 3.5, if $\dim X = 2$, then ϕ is a $\mathbb{P}_1$-bundle. If $\dim X = 3$ and if ϕ is birational, then $g : Y \to Z$ is étale of degree d and both ϕ and ψ are blow-ups of smooth curves.*

Proof. Recall that extremal loci of birational surface contractions are irreducible, simply connected divisors. This shows that X is minimal, that ϕ is of fiber type and settles the case $\dim X = 2$.

Now assume that $\dim X = 3$ and that ϕ is birational. Observe that if $\phi(E_R)$ was a point, the classification [Mor82, Thm. 3.3] yields that $E_{f_*(R)}$ is isomorphic to $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ or to the quadric cone. But all these spaces are simply connected, a contradiction. Consequence:

$\phi(E_R)$ is not a point, and [Mor82, Thm. 3.3] asserts that ϕ is a blow-up. The same holds for ψ, the contraction of $f_*(R)$. The fact that fibers of ψ are 1-connected yields the étalité of g. $\square$

3.B. **Extremal contractions in the presence of étale morphisms between non-isomorphic varieties.** We remark that the results of Proposition 3.2–Corollary 3.6 remain true for étale morphisms between possibly non-isomorphic varieties, as long as their second Betti numbers agree.

Proposition 3.7. *Let $g : X \to X'$ be a surjective étale morphism of degree d between projective manifolds that satisfy $b_2(X) = b_2(X')$. Then the following holds.*

(3.7.1) *The linear isomorphism of $\mathbb{R}$-vector spaces, $g_* : H_2(X, \mathbb{R}) \to H_2(X', \mathbb{R})$, restricts to a bijective map $g_* : \overline{NE}(X) \to \overline{NE}(X')$ and defines a bijection on the set of extremal rays.*

(3.7.2) *If $R \subset \overline{NE}(X)$ is an extremal ray, then there exists a commutative diagram as follows*

$$\begin{tikzcd} X \arrow[r, "\phi\text{, contraction of } R"] \arrow[d, "g\text{, étale}"'] & Y \arrow[d, "h\text{, finite}"] \\ X' \arrow[r, "\psi\text{, contraction of } f_*(R)"'] & Z. \end{tikzcd}$$

If E_R and $E_{g_*(R)}$ are the exceptional sets of ϕ and ψ, respectively, then $g(E_R) = E_{g_*(R)}$ and $E_R = g^{-1}(E_{g_*(R)})$. If the contraction ϕ is divisorial, then $g|_{E_R} : E_R \to E_{g_*(R)}$ is étale of degree d and $E_{g_*(R)}$ is thus not simply connected.

(3.7.3) *If $\dim X = 2$, then X is minimal. If $\dim X = 3$ and ϕ is birational, then h is étale of degree d and both ϕ and ψ are blow-ups of smooth curves.*

Proof. Since $g_* \circ g^* : H^*(X') \to H^*(X)$ is multiplication with $d = \deg(g)$, we have that $g^* : H^*(X') \to H^*(X)$ is (piecewise) injective. Since $b_2(X) = b_2(X')$, we deduce as in Lemma 2.2 that both $g^* : H^2(X') \to H^2(X)$ and $g_* : H^2(X) \to H^2(X')$ are isomorphic. The argumentation of Section 3.A can then be applied verbatim. $\square$

Proposition 3.7 will later be used in the following context. In the setup of Corollary 3.4, assume that $\phi : X \to Y$ is the blow-up of the projective manifold Y along a smooth curve C. Let $E = \phi^{-1}(C)$. It is then not difficult to see that $\psi : X \to Z$ is then also a blow-up of a manifold along a curve. Since $b_2(Y) = b_2(Z)$, Proposition 3.7 applies to contractions of Y.

4. Endomorphisms of non-uniruled varieties

In [Iit82, Thm. 11.7, p. 337] it was shown that endomorphisms of manifolds with $\kappa(X) \geq 0$ are necessarily étale. We generalize this to non-uniruled varieties, at least when X is projective. First we state the following weaker result which also holds for Kähler manifolds.

Theorem 4.1. *Let X be a compact Kähler manifold and $f : X \to X$ a surjective endomorphism. If K_X is pseudo-effective, i.e. if its class is in the closure of the Kähler cone, then f is étale.*

Proof. We argue by contradiction: assume that K_X is pseudo-effective and f not étale, i.e. assume that the ramification divisor of f is not trivial: $\mathfrak{R} \neq 0$. We fix a Kähler form ω on X. It is an immediate consequence of Lemma 2.2 that f is finite. The standard adjunction formula for a branched morphism, $K_X = f^*(K_X) + \mathfrak{R}$, then has two consequences:

First, the canonical bundle is not numerically trivial, $K_X \not\equiv 0$. Since K_X is assumed pseudo-effective, that means $K_X \cdot \omega^{n-1} > 0$ (in fact, the class of K_X is represented by a non-zero positive closed current T and it is a standard fact that $T \cdot \omega^{n-1} > 0$ unless $T = 0$).

Secondly, if $f_m := f \circ \cdots \circ f$ is the m^{th} iteration of f, the iterated adjunction formula reads

$$K_X = f_m^*(K_X) + f_{m-1}^*(\mathfrak{R}) + \ldots + f^*(\mathfrak{R}) + \mathfrak{R}.$$

Intersecting with ω^{n-1}, we obtain

$$K_X \cdot \omega^{n-1} = f_m^*(K_X) \cdot \omega^{n-1} + f_{m-1}^*(\mathfrak{R}) \cdot \omega^{n-1} + \ldots + \mathfrak{R} \cdot \omega^{n-1}. \tag{4.1.1}$$

Observe that there exists a number $c > 0$ such that $L \cdot \omega^{n-1} > c$ for all non-trivial pseudo-effective line bundles $L \in \mathrm{Pic}(X)$; the number c exists because the cohomology classes of pseudo-effective line bundles are exactly the integral points in the pseudo-effective cone. Since all of the $m + 1$ summands in equation (4.1.1) are therefore larger than c, we have

$$K_X \cdot \omega^{n-1} \geq (m + 1)c$$

for all positive integers m. This is absurd. □

Corollary 4.2. *Let X be a projective manifold, $f : X \to X$ surjective. If X is not uniruled, then f is étale.*

Proof. Since X is not uniruled, K_X is pseudo-effective by [BDPP04]. Now apply Theorem 4.1. □

Remark 4.3. If X is Kähler and $f : X \to X$ surjective but not étale, Theorem 4.1 asserts that K_X cannot be pseudo-effective. It is, however, unknown whether this implies that X is uniruled. For that reason we cannot state Corollary 4.2 in the Kähler case although we strongly believe that it will be true.

The proof of Theorem 4.1 shows a little more: if ω is any Kähler form and η any positive closed $(n-1, n-1)$-form, we have $K_X \cdot \omega^{n-1} \leq 0$ and $K_X \cdot \eta \leq 0$. Since $K_X \not\equiv 0$, we must actually have strict inequality for some η. This could be useful in a further study of the Kähler case.

5. ÉTALE ENDOMORPHISMS

In this section we study étale endomorphisms more closely. In Section 5.A, we consider varieties of arbitrary dimension, and study endomorphisms from a deformation-theoretic point of view. In Section 5.B we study the interaction of the endomorphism with the Albanese map. Next we restrict to dimension 3 and apply the minimal model program to X. Since the case of non-negative Kodaira dimension was treated by Fujimoto, we restrict ourselves to uniruled threefolds X.

Maintaining the Assumptions 2.1, we suppose throughout this section that f is étale. We set $n := \dim X$ and note that a number of invariants vanish.

Lemma 5.1. *In this setup, we have the following numerical data.*

(5.1.1) *The class $c_1(K_X)^n \in H^{2n}(X)$ is zero.*
(5.1.2) $\chi(\mathcal{O}_X) = 0$.

(5.1.3) $c_n(X) = 0$.

Proof. The first claim follows from $c_1(K_X)^n = f^*(c_1(K_X)^n) = d \cdot c_1(K_X)^n$. For the second claim, observe $\chi(\mathcal{O}_X) = d \cdot \chi(\mathcal{O}_X)$. The third results from $T_X = f^*(T_X)$. $\square$

5.A. **Deformations of étale endomorphisms.** This section is concerned with a study of deformations of f. We will show that all deformations of the étale morphism f come from automorphisms of X. This strengthens the results of [HKP06, KP05] in our case.

Theorem 5.2. *For an étale morphism $f : X \to X$, we have*

$$\mathrm{Hom}_f(X,X) \cong \mathrm{Aut}^\circ(X) \Big/ \mathrm{Aut}^\circ(X) \cap \mathrm{Aut}(X/X), \tag{5.2.1}$$

where $\mathrm{Aut}^\circ(X)$ is the maximal connected subgroup of $\mathrm{Aut}(X)$ and $\mathrm{Hom}_f(X,X)$ is the connected component of $\mathrm{Hom}(X,X)$ that contains f. In particular, $\mathrm{Hom}_f(X,X)$ is irreducible, reduced and smooth.

If X is not uniruled and if $h^0(X,T_X) > 0$, then there exists a finite étale cover $g : \tilde{X} \to X$ where $\tilde{X}$ is a product $\tilde{X} = A \times W$ such that A is a torus and such that $H^0(A,T_A) = g^* H^0(X,T_X)$.

Proof. Consider the quasi-finite composition morphism of quasi-projective schemes

$$\begin{array}{cccc} f^\circ: & \mathrm{Aut}^\circ(X) & \to & \mathrm{Hom}_f(X,X) \\ & g & \mapsto & f \circ g \end{array} \tag{5.2.2}$$

If we identify the tangent spaces $T_{\mathrm{Aut}^\circ(X)} \cong H^0(X,T_X)$, and $T_{\mathrm{Hom}_f(X,X)} \cong H^0(X, f^*(T_X))$, then the tangent morphism of f° at $Id \in \mathrm{Aut}^\circ(X)$, is simply the tangent map of f, i.e.,

$$Tf^\circ|_{Id} = H^0(Tf) : H^0(X,T_X) \to H^0(X, f^*(T_X)).$$

Since f is étale, $Tf^\circ|_{Id}$ is necessarily isomorphic. Using the group structure of $\mathrm{Aut}^\circ(X)$, the same argument yields that $Tf^\circ|_g$ is isomorphic for all $g \in \mathrm{Aut}^\circ(X)$. In particular, since $\mathrm{Aut}^\circ(X)$ is reduced and smooth, the image $U := f^\circ(\mathrm{Aut}^\circ(X))$ of the morphism is an open neighborhood of f in the Hom-scheme, and f° identifies U with the right hand side of (5.2.1). The bijectivity of the tangent map yields that $\mathrm{Hom}_f(X,X)$ is reduced and smooth along U.

It remains to show that f° is set-theoretically surjective. If not, let $f' \in \partial U \setminus U$ be a point in the boundary.

We claim that morphism $f' : X \to X$ is then again surjective and étale. For surjectivity, observe that the morphisms f and f' are homotopic, so that the associated pull-back morphisms on cohomology are equal. But a proper morphism surjective if and only if the pull-back of the orientation form is non-zero. For étalité, consider the morphism

$$\eta : \mathrm{Hom}_f(X,X) \to \mathrm{Pic}(X), \quad \text{where} \quad \eta(g) := g^*(K_X) - K_X.$$

Observe that the image of $\eta(g)$ is the trivial bundle iff g is étale, and that η is constant on the open set $U \subset \mathrm{Hom}_f(X,X)$.

Now f' being surjective and étale, we can again consider the composition morphism $(f')^\circ$, defined in analogy with (5.2.2). Its image $(f')^\circ(\mathrm{Aut}^\circ(X))$ is again open and therefore necessarily intersects U. This is to say that there are automorphisms $g, g' \in \mathrm{Aut}^\circ(X)$ such that

$$f \circ g = f' \circ g', \quad \text{that is,} \quad f' = f \circ (g \circ (g')^{-1}),$$

a contradiction to $f' \notin U$.

For the second statement, observe that a vector field on X cannot have a zero, because otherwise X would be uniruled. Hence [Lie78] gives the decomposition. □

5.B. **The Albanese map of a variety with étale endomorphisms.** We maintain the assumption that f is étale and study the Albanese map, see also [Fuj02].

Proposition 5.3. *Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese and $Y \subset \mathrm{Alb}(X)$ its image. Then f induces a finite étale cover $Y \to Y$.*

There exists a morphism $h : Y \to W$ to a variety of general type which is a torus bundle with fiber B and which is trivialized after finite étale cover of W. The map $W \to W$ induced by f is an automorphism.

Proof. The universal property of the Albanese implies that f induces an étale morphism $g : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ mapping Y to Y.

If $Y = \mathrm{Alb}(X)$, let W be a point. If $Y \neq \mathrm{Alb}(X)$, we obtain a map $h : Y \to W$ where W is of general type and the fibers are translates of subtori of $\mathrm{Alb}(X)$. Consider the induced map $g' : W \to W$, whose existence is guaranteed by a classical result of Ueno, see e.g. [Mor87, Thm. 3.7]. Since W is of general type, i.e., since any desingularization is of general type, g' is easily seen to be an automorphism — adapt the proof [Bea01, Prop. 2]. The fact that h can be trivialized by a finite étale cover of W is again Ueno's theorem cited above. □

5.C. **Extremal contractions of threefolds with étale endomorphisms.** As in [Fuj02, Sect. 4], we will now investigate étale morphisms of threefolds more closely. It will turn out without much work that any extremal contraction is the blow-up of a curve. More precisely, the following strengthening of Proposition 3.7 holds true.

Proposition 5.4. *Maintaining the Assumptions 2.1, suppose that X is a 3-fold and that f is étale of degree d. Then there exists a commutative diagram*

(5.4.1)
$$\begin{tikzcd} X \arrow[r, "\phi_0"] \arrow[d, "f"'] & Y_1 \arrow[r, "\phi_1"] \arrow[d, "h_1"] & Y_2 \arrow[r, "\phi_1"] \arrow[d, "h_2"] & \cdots \arrow[r, "\phi_{m-1}"] & Y_m \arrow[d, "h_m"] \\ X \arrow[r, "\psi_0"'] & Z_1 \arrow[r, "\psi_1"'] & Z_2 \arrow[r, "\psi_2"'] & \cdots \arrow[r, "\psi_{m-1}"'] & Z_m \end{tikzcd}$$

where the h_i are étale of degree d, the ϕ_i and ψ_i are extremal contractions and the following holds.

(5.4.2) *All Y_i and Z_i are smooth, ϕ and ψ are blow-ups along smooth elliptic curves on which the canonical bundle is numerically trivial.*

(5.4.3) *Either K_{Y_m} and K_{Z_m} are nef or Y_m and Z_m admit only contractions of fiber type.*

Proof. Applying Corollary 3.4 and Proposition 3.7 inductively, we obtain an infinite diagram, with (5.4.1) as the first two rows.

(5.4.4)
$$\begin{tikzcd}
X \arrow[r, "\phi_0"] \arrow[d, "f"'] & Y_1 \arrow[r, "\phi_1"] \arrow[d, "h_{1,0}"] & Y_2 \arrow[r, "\phi_1"] \arrow[d, "h_{2,0}"] & \cdots \arrow[r, "\phi_{m-1}"] & Y_m \arrow[d, "h_{m,0}"] \\
X \arrow[r, "\psi_{0,1}"'] \arrow[d, "f"'] & Z_{1,1} \arrow[r, "\psi_{1,1}"'] \arrow[d, "h_{1,1}"] & Z_{2,1} \arrow[r, "\psi_{2,1}"'] \arrow[d, "h_{2,1}"] & \cdots \arrow[r, "\psi_{m-1,1}"'] & Z_{m,1} \arrow[d, "h_{m,1}"] \\
X \arrow[r, "\psi_{0,2}"'] \arrow[d, "f"'] & Z_{1,2} \arrow[r, "\psi_{1,2}"'] \arrow[d, "h_{1,2}"] & Z_{2,2} \arrow[r, "\psi_{2,2}"'] \arrow[d, "h_{2,2}"] & \cdots \arrow[r, "\psi_{m-1,2}"'] & Z_{m,2} \arrow[d, "h_{m,2}"] \\
\vdots & \vdots & \vdots & & \vdots
\end{tikzcd}$$

The étalité of the $h_{i,j}$, the smoothness of the $Z_{i,j}$ and fact that the $\psi_{i,j}$ are blow-ups of smooth curves $D_i \subset Y_i$ and $C_{i,j} \subset Z_{i,j}$ are immediate from Proposition 3.7. Statement (2) of Proposition 5.4 is a standard result of minimal model theory.

It remains to show that the curves D_i and $C_{i,j}$ are elliptic. To this end, observe that the $h_{i,j}$ restrict to étale morphisms of the $C_{i,j}$, of degree d. Thus, for any number i and j,

$$\deg K_{D_i} = d^j \cdot \deg K_{C_{i,j}} \text{ and } \deg K_{Y_i}|_{D_i} = d^j \cdot \deg K_{Z_{i,j}}|_{C_{i,j}}.$$

This is possible only if $\deg K_{D_i} = \deg K_{Y_i}|_{D_i} = 0$. The same argument holds for any $C_{i,j}$. □

Notation 5.5. In the setup of Proposition 5.4, we call $h_m : Y_m \to Z_m$ a *minimal model* of f.

5.D. **The minimal model program if $\kappa(X) = -\infty$.** Since the case $\kappa(X) \geq 0$ was studied in [Fuj02] in great detail, we are mainly interested in the case where $\kappa(X) = -\infty$. In this setup the manifolds Y_m and Z_m of Proposition 5.4 allow extremal contractions of fiber type. To fix notation, we summarize the obvious properties in the following Lemma.

Lemma 5.6. *Maintaining the assumptions of Proposition 5.4, suppose that $\kappa(X) = -\infty$. Then the commutative diagram* (5.4.1) *extends as follows.*

$$\begin{tikzcd}
X \arrow[r, "\phi_0"] \arrow[d, "f"'] & Y_1 \arrow[r, "\phi_1"] \arrow[d, "h_1"] & Y_2 \arrow[r, "\phi_1"] \arrow[d, "h_2"] & \cdots \arrow[r, "\phi_{m-1}"] & Y_m \arrow[r, "\sigma"] \arrow[d, "h_m"] & S \arrow[d, "h"] \\
X \arrow[r, "\psi_0"'] & Z_1 \arrow[r, "\psi_1"'] & Z_2 \arrow[r, "\psi_2"'] & \cdots \arrow[r, "\psi_{m-1}"'] & Z_m \arrow[r, "\tau"'] & T
\end{tikzcd}$$

where the σ and τ are contractions of fiber type with $\rho(Y_m/S) = \rho(Z_m/T) = 1$ and h is finite with $\deg h = \deg f$.

Proof. The existence of the diagram follows again from Proposition 3.7. The fact that the general fiber of τ is Fano, hence simply connected, implies that $\deg h = \deg f$. □

In Sections 5.D.1 and 5.D.2 we will consider the cases where $\dim S$ is 1 or 2 separately. It will turn out in either case that $S \cong T$ and that h is étale.

5.D.1. *Minimal models over curves.*

Proposition 5.7. *In the setup of Lemma 5.6, suppose that* $\dim S = 1$. *Then* $S \cong \mathrm{Alb}(X)$, h *is étale and the fibration* σ *is locally trivial in the analytic topology. The* ϕ_i *are blow-ups of elliptic curves which are multi-sections over* S.

Proof. Apply Proposition 3.7 to the two leftmost columns of Diagram (5.4.4) and obtain

$$\begin{tikzcd} Y_m \arrow[r, "h_{m,0}"] \arrow[d, "\sigma"'] & Z_{m,1} \arrow[r, "h_{m,1}"] \arrow[d, "\tau_1"] & Z_{m,2} \arrow[r, "h_{m,2}"] \arrow[d, "\tau_1"] & \cdots \\ S \arrow[r, "h'_0"'] & T_1 \arrow[r, "h'_1"'] & T_2 \arrow[r, "h'_2"'] & \cdots \end{tikzcd}$$

where T_1 are curves and τ_1 are contractions of fiber type. Again, all h'_i are finite of degree d.

As the general fiber of σ is rationally connected, it is clear that $q(X) = q(Y_m) = g(S)$. Observe that $g(S) \geq 1$: if not, $S \cong \mathbb{P}_1$, and the theorem of Graber-Harris-Starr [GHS03] would imply that X is rationally connected, hence simply connected, a contradiction. The same argumentation shows $g(T_i) \geq 1$ for all i. But then

$$\deg K_S \geq d \cdot \deg K_{T_1} \geq \cdots \geq d^i \cdot \underbrace{\deg K_{T_i}}_{\geq 0}$$

for all i, which is possible if and only if S and all T_i are elliptic, and all h'_i étale.

Since the general fiber of σ is Fano, it follows that the Albanese map $X \to \mathrm{Alb}(X)$ factors via S. Since fibers of $X \to S$ are connected and S is already elliptic, S is $\mathrm{Alb}(X)$. The same holds for any of the T_i.

To show that σ is locally trivial, choose an arbitrary point $s \in S$ and observe that for any i and any point of s' of

$$F_i := (h'_i \circ \cdots \circ h'_0)^{-1}(h'_i \circ \cdots \circ h'_0(s)),$$

the scheme-theoretic fibers $\sigma^{-1}(s)$ and $\sigma^{-1}(s')$ are isomorphic. Since the cardinality $\#F_i = d^i$ becomes arbitrarily large, this shows that all scheme-theoretic σ-fibers are isomorphic. In particular, σ is smooth and thus, by [Kod86, Thm. 4.2], locally trivial in the analytic topology.

The description of ϕ is immediate from Proposition 5.4. □

Remark 5.8. If $\sigma : Y_m \to S$ is not a $\mathbb{P}_2$-bundle, we can say a bit more about its structure.

First suppose that the fibers F of σ are proper del Pezzo surfaces, i.e., $K_F^2 \leq 6$. Then we can apply [PS98, Prop. 0.4] and obtain a finite étale cover $\tilde{S} \to S$ such that the fiber product $\tilde{Y}_m = Y_m \times_S \tilde{S}$ contains a divisor D such that $D \to \tilde{S}$ is a $\mathbb{P}_1-$bundle and $D \cap \tilde{F}$ is a (-1)-curve for all fibers $\tilde{F}$ of $\tilde{Y}_m \to \tilde{S}$. Then we can consider the resulting space $\tilde{Y}_{m,1} \to \tilde{S}$ and repeat the process, ending up with a $\mathbb{P}_2-$bundle over an étale cover of S.

If σ is $\mathbb{P}_1 \times \mathbb{P}_1-$bundle, then we find a finite étale cover $\tilde{S} \to S$ such that $\tilde{Y}_m$ has a $\mathbb{P}_1-$bundle structure of a surface W which is in turn again a $\mathbb{P}_1-$bundle over $\tilde{S}$.

5.D.2. *Minimal models over surfaces.* We are next studying the case $\dim S = 2$. As in case $\dim S = 1$, the morphism h will turn out to be étale.

Lemma 5.9. *In the setup of Lemma 5.6, suppose that* $\dim S = \dim T = 2$. *Then the morphism* h *is étale of degree* d.

Proof. It suffices to observe that all fibers of τ are conics, hence simply connected. □

Proposition 5.10. *In the setup of Lemma 5.6, suppose that* $\dim S = \dim T = 2$ *and that* $\kappa(S) \geq 0$. *Then* $S \cong T$. *A finite étale base-change makes* Y_m *a conic bundle over an abelian surface or over a product* $C \times E$ *with* E *elliptic curve, and* C *a curve of genus at least two. In particular, in the second case* S *is an étale quotient of the product of an elliptic curve with a curve of general type.*

The discriminant locus of the conic bundle σ *is either empty or a disjoint union of elliptic curves.*

Proof. The morphisms σ and τ are conic bundles by Mori theory. We observe that $S \cong T$, because both are mapped isomorphically onto the same irreducible component of the cycle space of X. In fact, otherwise X would have two different 2-dimensional families of rational curves. This would imply that S is uniruled, hence $\kappa(S) = -\infty$.

Since h is not an isomorphism, [Fuj02, Thm. 3.2] shows that there are two distinct cases:

(5.10.1) $\kappa(S) = 0$ and S is abelian or hyperelliptic;

(5.10.2) $\kappa(S) = 1$ and after finite étale cover, $S = E \times C$ with E elliptic and $g(C) \geq 2$.

It remains to prove the last statement. Let $\Delta_S \subset S$ be the discriminant locus of σ. Again the fact that f is étale implies that $h^{-1}(\Delta_S) = \Delta_S$. Thus Δ_S is either empty or a disjoint union of elliptic curves. □

Proposition 5.11. *Suppose* $\kappa(X) = -\infty$ *and that* S *and* T *are surfaces with* $\kappa(S) = \kappa(T) = -\infty$. *Then* S *and* T *are ruled surfaces over an elliptic curve* B *and one of the following two situations occurs.*

(5.11.1) *The minimal model* Y_m *is a proper conic bundle, and the discriminant locus consists of étale multi-sections of* $S \to B$. *Supposing moreover that* X *and all blow-downs* Y_i *have only finitely many extremal rays, or that already* X *is minimal, then, replacing* f *by an iterate* f_k *if necessary, we obtain* $Y_m = Z_m$ *and* $S = T = \mathbb{P}(\mathcal{F})$ *with* $\mathcal{F}$ *a semi-stable rank 2-bundle over* B.

(5.11.2) *The minimal model* Y_m *is of the form* $Y_m = \mathbb{P}(\mathcal{E})$ *with* $\mathcal{E}$ *a rank 2-vector bundle over* S *with* $c_1(\mathcal{E})^2 = 4c_2(\mathcal{E})$. *In this case, one of the following two conditions is satisfied:*
   (1) *for all ample classes* H, *the bundle* $\mathcal{E}$ *is not* H*-stable, or*
   (2) $Y_m = S \times_B \mathbb{P}(\mathcal{E}')$ *with* $\mathcal{E}'$ *a rank 2-bundle over* B, $T = \mathbb{P}(\mathcal{E}')$ *and* $\mathcal{E} = p^*(\mathcal{E}') \otimes \mathcal{L}$ *where* $\mathcal{L} \in \mathrm{Pic}(S)$, *and* $p : S \to B$ *denotes the projection.*

Proof. Step 1. Since S and T have both negative Kodaira dimension and since none of them can be rational (otherwise X would be rationally connected, hence simply connected), S and T are birationally ruled over curves B respectively, C of positive genus. Let $p : S \to B$ resp. $q : T \to C$ denote the projections. Then we observe that

$$q(X) = q(Y_m) = q(S) = g(B) \quad \text{and} \quad q(X) = q(Z_m) = q(T) = g(C).$$

The étale morphism $h : S \to T$ from Lemma 5.6 immediately yields an étale morphism $h' : B \to C$ of degree $d \geq 2$. From Riemann-Hurwitz we deduce that B and C must be elliptic. Hence the composed maps $\alpha : X \to B$ and $\beta : X \to C$ coincide both to the Albanese map of X, which shows that $B = C$. As in the proof of Proposition 5.7, the étalité of h' then immediately implies that p and q are submersions. Consequently S and T are minimal, i.e., ruled surfaces over the elliptic curve B.

Step 2. Consider the case of (5.11.1) and suppose now that the discriminant locus $\Delta_S \neq \emptyset$. So does $\Delta_T \neq \emptyset$ and both are disjoint unions of elliptic curves, i.e., multi-sections of p and

q. Assume that X and all Y_i have only finitely many extremal rays. Then $\overline{NE}(X/B)$ has only finitely many extremal rays. Hence we may pass to an iterate f_k such that $Y_1 = Z_1$. Now we argue with h_1 instead of f and proceed inductively to conclude $Y_m = Z_m$. A last application of this argument yields $S = T$, and, by Lemma 5.12 below, S is defined by a semi-stable vector bundle. This shows Claim (5.11.1).

Step3. We now consider case (5.11.2) and suppose that $\Delta_S = \Delta_T = \emptyset$. Thus both $\sigma : Y_m \to S$ and $\tau : Z_m \to T$ are $\mathbb{P}_1-$bundles. We have to distinguish two cases:

(A) Y_m carries only one $\mathbb{P}_1-$bundle structure, so $S = T$, or
(B) Y_m carries two $\mathbb{P}_1-$ bundle structures.

In both cases we can write $Y_m = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2-bundle over S. An explicit computation of Chern classes, using that $(K_{Y_m})^3 = 0$ and $(K_S)^2 = 0$, implies

$$c_1(\mathcal{E})^2 = 4c_2(\mathcal{E}). \tag{5.11.1}$$

In the case (A), if $\mathcal{E}$ is stable with respect to some ample H, then Equation (5.11.1) and [AB97] or [Tak72, Thm. 3.7] imply that $\mathcal{E} = p^*(\mathcal{E}') \otimes \mathcal{L}$, where $\mathcal{E}'$ is a rank-2 bundle over B, and $\mathcal{L}$ is a line bundle on S. In particular, any fiber of $Y_m \to B$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and Y_m has two different $\mathbb{P}_1$-bundle structures contrary to our assumption.

In case (B), any fiber F of $Y_m \to B$ has two rulings, hence $F \simeq \mathbb{P}_1 \times \mathbb{P}_1$. If $b \in B$ is any point and S_b the associated fiber, we can therefore normalize $\mathcal{E}$ such that $\mathcal{E}|_{S_b} = \mathcal{O}_{S_b}^2$. It is then possible to write $\mathcal{E} = p^*(\mathcal{E}')$ with a rank-2 bundle $\mathcal{E}'$ over B. Hence

$$Y_m = S \times_B \mathbb{P}(\mathcal{E}')$$

and $T = \mathbb{P}(\mathcal{E}')$. The étale map $h : S \to T$ means that writing $S = \mathbb{P}(\mathcal{G})$, we have $\mathcal{G} = j^*(\mathcal{E}')$ up to a twist with a line bundle. This shows (5.11.2) and ends the proof. □

We prove next the technical result in dimension 2 which was used above.

Lemma 5.12. *Let $F := \mathbb{P}(\mathcal{F}) \to E$ be a ruled surface over an elliptic curve. If S has a non-trivial étale endomorphism, then $\mathcal{F}$ is semi-stable.*

Proof. We know from [Nak02] that F has a non-trivial endomorphism. Moreover, if $\mathcal{F}$ is indecomposable, then it was proved again in [Nak02] that F has an étale endomorphism. We have thus to analyze the decomposable case. After normalization [Har77], we can assume $\mathcal{F} \cong \mathcal{O}_E \oplus L$, with $\deg(L) \leq 0$. Then $\mathcal{F}$ is semi-stable if and only if $\deg(L) = 0$. If S has an étale endomorphism, it must have degree one on the fibers, and thus it comes from an endomorphism ψ of E. The compatibility condition is that $\psi^*\mathcal{F} \cong \mathcal{F} \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}(E)$. This is possible only if either $\mathcal{L} \cong \mathcal{O}_E$ and $\psi^* L \cong L$, or $\mathcal{L} \cong L^{-1}$ and $\psi^* L \cong L^{-1}$. In both cases, we obtain $\deg(L) = 0$. The decomposable case really occurs, as exemplified by the trivial 2-bundle on E. □

5.D.3. *Proper conic bundles with étale endomorphisms.* In this part we give non-trivial examples of proper conic bundles with étale endomorphisms. The rough construction idea is the following: it is clear that there are conic bundles with large relative Picard number having endomorphisms. We start with such a conic bundle, then we try the drop the second Betti number by factorization.

Proposition 5.13. *Let B be an arbitrary curve, and E be an elliptic curve, and denote $Y = E \times B$. Let k be an odd positive integer. Then there exists a smooth threefold X and a morphism $\phi : X \to Y$ with $\rho(X/Y) = 1$, which realizes X as a proper conic bundle with*

reduced, but not always irreducible fibers. Further, there exists an étale endomorphism f of X of degree k^2 making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

where h denotes the multiplication by k on the first factor.

Proof. Step 1: As a first step, we construct a proper conic bundle $S \to B$ with reduced, but not always irreducible fibers that carries a B-involution which interchanges the components of the reducible fibers.

Let D be an effective, reduced divisor on the curve B such that $\mathcal{O}_B(D)$ is divisible by two in Pic(B). Consider $L \in$ Pic(B) with $L^{\otimes 2} = \mathcal{O}_B(D)$ and $s \in H^0(B, \mathcal{O}_B(D))$ such that s vanishes precisely along D. Denote $\mathbb{L}$ the total bundle space of L. It is known that the subvariety

$$C := \{x \in \mathbb{L}, x^{\otimes 2} = s\}$$

is a smooth double covering of B under the restriction of the bundle projection $\mathbb{L} \xrightarrow{\pi} B$. The intersection between the inverse image of D under π and C coincides set-theoretically with the ramification divisor D' of the covering $C \to B$.

We compactify $\mathbb{L}$ to the ruled surface over B

$$S := \mathbb{P}(\mathcal{O}_B \oplus L),$$

where the projectivization is taken in the usual geometric sense, opposite to Grothendieck's. We keep the notation π for the projection $S \to B$. The natural morphism between vector bundles

$$\begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} : \mathcal{O}_B \oplus L \to L \oplus L^{\otimes 2}$$

yields to a rational involution $S \dashrightarrow S$ which is relative over B and whose indeterminacy locus is the ramification divisor D' of $C \to B$. We consider the blow-up $\widetilde{S}$ of S in the points of D' so that the rational involution of S lifts to a morphism $\widetilde{S} \to S$. Note that the strict transforms of the fibers through the points of D' will be contracted by this morphism, a fact which eventually proves that the original rational involution factors through a regular involution $\iota : \widetilde{S} \to \widetilde{S}$. By construction, the fixed locus of the involution ι coincides with the strict transform of C, and ι exchanges the irreducible components of each of the inverse images of fibers through the points of D'.

Step 2: Now X is constructed as a suitable quotient of $S \times E$. We write the elliptic curve E as $E = \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z} + \mathbb{Z}.\omega$, and we identify the involution ι with a 2-torsion element of E, say $1/2$. Via the canonical projection $E[2] \to \langle \iota \rangle$, $1/2 \mapsto \iota$, $\omega/2 \mapsto id$, the group $E[2] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of all 2-torsion elements of E acts diagonally on the product $E \times \widetilde{S}$. Since the action of $E[2]$ on E by translations is with trivial isotropy groups, so is the action of $E[2]$ on the product $E \times \widetilde{S}$, hence the quotient

$$X := (E \times \widetilde{S})/E[2]$$

is a smooth variety.

Consider next the group of $2k$-torsion elements $E[2k]$, and note that, since k is odd, the inclusion $E[2] \subset E[2k]$ gives a decomposition $E[2k] \cong E[2] \times E[k]$. Similarly to above,

we identify ι with $1/2 \in E[2k]$, and consider the diagonal action of $E[2k]$ on the product $E \times \widetilde{S}$. The action of the component $E[k]$ on $\widetilde{S}$ is trivial, and we observe that

$$(E \times \widetilde{S})/E[2k] \cong X.$$

The inclusion $E[2] \subset E[2k]$ yields to a $(k^2 : 1)$ étale covering

$$(E \times \widetilde{S})/E[2k] \to (E \times \widetilde{S})/E[2]$$

that descends to the natural $(k^2 : 1)$ covering

$$E \times B \cong E/E[2k] \times B \to E/E[2] \times B \cong E \times B.$$

which is given by multiplication with k. It is elementary to check that all components of ϕ-fibers have linearly dependent homology classes. The assertion that $\rho(X/Y) = 1$ follows. $\square$

The vector bundle associated with an endomorphism

6. Positivity of vector bundles associated to Galois coverings

This section is devoted to the study of positivity properties of vector bundles coming from Galois covers. The Galois condition will be used in the following form.

Lemma 6.1. *Let $f : Y \to X$ be a finite morphism of degree d between irreducible reduced complex spaces. Then f is Galois if and only if the fibered product $Z := Y \times_X Y$ decomposes as follows*

$$Y \times_X Y = \bigcup_{j=1}^{d} Z_j, \tag{6.1.1}$$

where the restriction of the first projection to any of the Z_j is biholomorphic to Y.

Proof. The proof is rather straightforward, and very likely well-known. We notice first that the image of the diagonal map $Y \to Y \times_X Y$ is one component Z_1 of $Y \times_X Y$ which projects isomorphically to Y. Any automorphism $\sigma_i \in G(Y/X)$ naturally induces an automorphism in $\widetilde{\sigma}_j \in G\big((Y \times_X Y)/Y\big)$, acting on the second factor. The image $Z_j := \widetilde{\sigma}_j(Z_1)$ is then an irreducible component of $Y \times_X Y$ which projects biholomorphically to Y.

If f is Galois of degree d, then we get precisely d such components. Since the degree of the first projection also equals d, we obtain a decomposition as in (6.1.1).

Conversely, if Z decomposes as in (6.1.1), we can define elements in $G(Y/X)$ using that all components Z_j are isomorphic. $\square$

Notation 6.2. Given any finite flat morphism $f : Y \to X$ with smooth target X, we consider the vector bundle

$$\mathcal{E}_f := \Big(f_*(\mathcal{O}_Y) \Big/ \mathcal{O}_X \Big)^*.$$

Recall that the trace map gives a splitting $f_*(\mathcal{O}_Y) \cong \mathcal{O}_X \oplus \mathcal{E}_f^*$.

To define $\mathcal{E}_f$ it is a priori it is not necessary to make any assumption on the smoothness of Y, nor of its components. Moreover, flatness is preserved in some cases when components of Y are removed.

Lemma 6.3. *Let Z be a reduced projective variety and $f : Z \to Y$ a finite flat morphism, with smooth irreducible projective target Y. If $Z = \bigcup_{j=1}^{d} Z_j$ is the decomposition into irreducible components, and $Z' := \bigcup_{j=2}^{d} Z_j$, then the restriction $f' := f|_{Z'}$ is likewise flat.*

Proof. We need to prove that $f'_*(\mathcal{O}_{Z'})$ is locally free, or equivalently that the length $l(Z'_y)$ of all scheme-theoretic fibers $Z'_y := (f')^{-1}(y)$ are the same. We shall proceed by induction on $n = \dim Y$.

If $n = 1$, flatness is equivalent to Y being dominated by any irreducible component of Z', [Har77, III Prop. 9.7]. This condition is fulfilled.

In general, take any point $y \in Y$ and consider a smooth connected hyperplane section $H \subset Y$ passing through y. The restriction $f|_H : Z_H \to H$ is obviously flat, and the restriction $f'|_H : Z'_H \to H$ is flat by induction hypothesis. Observe that $Z_H = \bigcup_j Z_{j,H}$, and that $Z_{j,H}$ surjects to H. Hence

$$l(Z'_{H,x}) = \deg(f'_H) = \deg(f').$$

Since $Z'_x = Z'_{H,x}$, we conclude. □

Theorem 6.4. *Let $Z = \bigcup_{j=1}^{d} Z_j$ be a connected reduced projective variety of dimension at least 2, where Z_j denote its irreducible components and $d \geq 2$. Suppose that for all $k \neq j$, the intersection $Z_j \cap Z_k$ is either empty, or an ample Cartier divisor in Z_j.*

If $f : Z \to Y$ is a finite flat morphism to a smooth variety such that for any j, the restriction $f|_{Z_j} : Z_j \to Y$ is biholomorphic, then $\mathcal{E}_f$ is ample.

Proof. We proceed by induction on the degree $d = \deg(f) \geq 2$.

After renumeration of the irreducible components Z_j, we can assume without loss of generality that $Z' = \bigcup_{j \geq 2} Z_j$ is connected. Write $Z = Z_1 \cup Z'$, and let $\mathfrak{R} \subset Z_1$ the scheme-theoretic intersection, $\mathfrak{R} := Z_1 \cap Z'$. Since $\mathfrak{R}$ is a union of ample divisors, it is ample.

From Lemma 6.3, it follows that the induced map from $f' : Z' \to Y$ is again flat. The Mayer-Vietoris sequence of the decomposition then reads as follows

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z'} \to \mathcal{O}_{\mathfrak{R}} \to 0.$$

Taking f_*, taking quotients by $\mathcal{O}_Y$ and using that Z_1 projects biholomorphically to Y, we obtain

$$0 \to \mathcal{E}_f^* \to \mathcal{O}_Y \oplus \mathcal{E}_{f'}^* \xrightarrow{\beta} \mathcal{O}_{\mathfrak{R}} \to 0.$$

Remark that $\beta|_{\mathcal{O}_Y \oplus \{0\}} : \mathcal{O}_Y \to \mathcal{O}_{\mathfrak{R}}$ is the obvious restriction map, so that

$$\ker\left(\beta|_{\mathcal{O}_Y \oplus \{0\}}\right) \cong \mathcal{O}_Y(-\mathfrak{R}).$$

The Snake Lemma then yields a short exact sequence

$$0 \to \mathcal{O}_Y(-\mathfrak{R}) \to \mathcal{E}_f^* \to \mathcal{E}_{f'}^* \to 0. \tag{6.4.1}$$

If $d = 2$, then $\mathcal{E}_{f'} = 0$, and Sequence (6.4.1) gives an isomorphism $\mathcal{E}_f \cong \mathcal{O}_Y(\mathfrak{R})$. Since the latter is ample, This settles the first induction step.

If $d \geq 3$, the morphism f' obviously satisfies all assumptions of Theorem 6.4, and $\mathcal{E}_{f'}$ is ample by induction hypothesis. The dual of Sequence (6.4.1) then represents $\mathcal{E}_f$ as an extension of ample bundles. It is thus ample, too. □

An almost identical argument, applied under weaker assumptions gives rise to a nefness criterion.

Proposition 6.5. *Let $Z = \bigcup_{j=1}^{d} Z_j$ be a reduced projective variety of dimension at least 2, where Z_j denote its irreducible components and $d \geq 2$. Suppose that for all $k \neq j$, the intersection $Z_j \cap Z_k$ is either empty, or a nef Cartier divisor in Z_j.*

If $f : Z \to Y$ is a finite flat morphism to a smooth variety such that for any j, the restriction $f_j := F|_{Z_j} : Z_j \to Y$ is biholomorphic, then $\mathcal{E}_f$ is nef. □

In view of Lemma 6.1, Theorem 6.4 applies to fibered products. The setup is the following. Suppose $f : Y \to X$ is a finite Galois covering (of degree at least two) of projective manifolds. Consider $Z = Y \times_X Y$, and denote $g : Z \to Y$ the first projection. Then $\mathcal{E}_f$ is ample if and only if $f^*\mathcal{E}_f$ is ample. An obvious base-change formula gives $f^*\mathcal{E}_f = \mathcal{E}_g$. Lemma 6.1 applies, and we obtain a decomposition $Z = \bigcup_{j=1}^{d} Z_j$, with all Z_j isomorphic to Y. Connectivity of Z is studied in the following.

Lemma 6.6. *With the notation of Lemma 6.1, the variety Z is connected if and only if f does not factor through $Y \to \widetilde{Y} \xrightarrow{\tilde{f}} X$, with $\tilde{f} : \widetilde{Y} \to X$ is étale of degree at least two.*

Proof. Suppose first that f factors through an étale covering as above. Then [Gro67, p. 63, thm 17.4.1] shows that $\widetilde{Y} \times_X \widetilde{Y}$ is not connected, as the diagonal is one of its connected components. Since Z obviously dominates $\widetilde{Y} \times_X \widetilde{Y}$, it cannot be connected.

Conversely, suppose that Z is not connected. Decompose Z into its connected components, $Z = \widetilde{Z}_1 \cup \cdots \cup \widetilde{Z}_p$. We can assume that the diagonal Δ is contained in $\widetilde{Z}_1$. Denote furthermore $\widetilde{Z}' = \widetilde{Z}_2 \cup \cdots \cup \widetilde{Z}_p$.

As in the proof of Lemma 6.1, we will identify an element in the Galois group $G(Y/X)$ with its induced automorphism of Z without further mention. Consider the stabilizer of $\tilde{Z}_1$ in the Galois group,

$$H := \{\sigma \in G(Y/X) \mid \sigma(\widetilde{Z}_1) = \widetilde{Z}_1\}.$$

The action of H on Z gives rise to an action of H on $\widetilde{Z}_1$. It is moreover clear that $\sigma(\widetilde{Z}') = \widetilde{Z}'$, for any $\sigma \in H$, so that H acts also on $\widetilde{Z}'$.

Set $\widetilde{Y} := Y/H$. Since Y is irreducible, we obtain that $\widetilde{Y}$ is irreducible, too. We claim that the induced morphism $\tilde{f} : \widetilde{Y} \to X$ is étale, in particular, we claim that $\widetilde{Y}$ is smooth. The proof is finished if this claim is shown.

Following [Gro67, Thm. 17.4.1] again, to prove the claim, it suffices to prove that the diagonal $\Delta_{\tilde{Y}} \subset \widetilde{Y} \times_X \widetilde{Y}$ is a connected component of $\widetilde{Y} \times_X \widetilde{Y}$. To this end, remark that

$$\widetilde{Y} \cong \widetilde{Z}_1 \big/ H, \quad \text{and} \quad \widetilde{Y} \times_X \widetilde{Y} \cong Z \big/ H \cong \widetilde{Z}_1 \big/ H \cup \widetilde{Z}' \big/ H.$$

Since $\widetilde{Z}_1$ and $\widetilde{Z}'$ are disjoint and open, we obtain that $\widetilde{Z}_1/H$ and $\widetilde{Z}'/H$ disjoint and open in the quotient topology. Since $\Delta_{\widetilde{Y}} = \widetilde{Z}_1/H$, we obtain the claim. □

We arrive at the main results of this section.

Theorem 6.7. *Let $f : Y \to X$ be a Galois covering of smooth varieties which does not factor through an étale covering of X, such that all irreducible components of the ramification divisor $\mathfrak{R}$ are ample on Y. Then the bundle $\mathcal{E}_f$ is ample.*

Proof. Recall from Lemma 6.6 that $Y \times_X Y$ is connected. Now apply Theorem 6.4. □

Corollary 6.8. *Let $f : Y \to X$ be a Galois covering of degree at least two of smooth projective varieties with $\pi_1(X) = 0$, and $\rho(Y) = 1$. Then the bundle $\mathcal{E}_f$ is ample.* □

The following example shows that the Galois condition in Corollary 6.8 is really necessary.

Example 6.9. *In* [PS04, Example 2.1] *an example of a triple covering $f : Y \to X$ of Fano threefolds with $\rho = 1$ is established with the property that $\mathcal{E}_f$ is not ample. Hence f cannot be Galois; moreover the Galois group $G(Y/X)$ must be trivial, since f cannot factor.*

Proposition 6.10. *Let $f : Y \to X$ be a Galois covering of smooth varieties, such that all the irreducible components of the ramification divisor $\mathfrak{R}$ are nef on Y and such that $\mathfrak{R}$ is reduced. Then the bundle $\mathcal{E}_f$ is nef.* □

7. BRANCHED ENDOMORPHISMS

In this section, we study endomorphisms $f : X \to X$ with non-empty branch locus. By Corollary 4.2, we know that X is uniruled. If $\rho(X) > 1$, one has to study the effect of f onto a Mori contraction. Here, however, we consider the case where $\rho(X) = 1$, and X is therefore Fano. We wish to address the following well-known problem, at least under suitable assumptions.

Problem 7.1. Is $\mathbb{P}_n$ the only Fano manifold with $\rho(X) = 1$ admitting an endomorphism of degree $\deg f > 1$?

There are a number of cases where Problem 7.1 has a positive answer.

- X is 3-dimensional: [ARVdV99, Sch99, HM03].
- X is rational homogeneous: [PS89, HM99].
- X is toric: [OW02].
- X contains a rational curve with trivial normal bundle: [HM03, Cor. 3].

In this section, we give a positive answer in the following cases.

- f is Galois and the branch locus $\mathfrak{B}$ is not stabilized by the action of $\mathrm{Aut}^0(X)$: Corollary 7.11 and Proposition 7.12.
- f is Galois, X is almost homogeneous and $h^0(X, T_X) > n$: Corollaries 7.11 and 7.16, Remark 7.16.1.
- f is Galois and X has a holomorphic vector field whose zero locus is not is not contained in the union of the branch loci of all the iterates of f: Corollary 7.11 and Theorem 7.13.

Here the condition that f is Galois is only used to ensure that the bundle $\mathcal{E}_f$ is ample. In other cases we can guarantee that any endomorphism $f : X \to X$ must have degree one.

- X is of index $r \leq 2$, and has a line ℓ not contained in the branch locus R of f: Theorem 7.6.
- X satisfies the *Cartan-Fubini Property* from Definition 7.17, X is almost homogeneous and $h^0(X, T_X) > n$: Theorem 7.19.

Numerous sub-cases, smaller results and variants are emphasized in the text.

7.A. Notation and assumptions.

In addition to the Assumptions 2.1, we fix the following extra assumptions and notation throughout the present Section 7.

Assumption 7.2. Assume that X is a Fano manifold of dimension $\dim X = n$, Picard number $\rho(X) = 1$ and index r. We maintain the assumption that there exists an endomorphism $f : X \to X$ of degree $d \geq 2$.

Notation 7.3. Denote the ample generator of the Picard group by $\mathcal{O}_X(1)$ and write

$$f^*\big(\mathcal{O}_X(1)\big) = \mathcal{O}_X(\mu).$$

We shall again consider the vector bundle

$$\mathcal{E} = \mathcal{E}_f = \Big(f_*(\mathcal{O}_X) \Big/ \mathcal{O}_X \Big)^*,$$

that was already introduced in Notation 6.2. Recall that $\operatorname{rank} \mathcal{E} = \deg(f) - 1 = d - 1$.

If $k \in \mathbb{N}$ is any number, let $f_k = f \circ \cdots \circ f$ denote the kth iterate of f, and let $\mathcal{E}_{f_k}$ be the associated vector bundle.

Remark 7.4. Equation (2.1.1) immediately yields

$$\mathcal{O}_X(\mathfrak{R}) = (1 - \mu) \cdot K_X. \tag{7.4.1}$$

Remark 7.5. If $Z \subset X$ is a subvariety of pure dimension, define its degree as $\deg Z = Z.c_1\big(\mathcal{O}_X(1)\big)^{\dim Z}$. Comparing $c_1(\mathcal{O}_X(1))$ with its pull-back, a standard computation of intersection numbers yields

$$\deg f^*(Z) = \mu^{\dim X - \dim Z} \cdot \deg Z. \tag{7.5.1}$$

In particular, if Z is a point, we obtain $d = \mu^{\dim X}$.

7.B. **Fano manifolds with small index.** We begin with the first case which is independent from the rest of the Section. The proof uses methods from [Sch99].

Theorem 7.6. *Let X be Fano with $\rho(X) = 1$ and index $r \leq 2$. Let $f : X \to X$ be an endomorphism. Suppose that there is a line $\ell \subset X$ not contained in the branch locus $\mathfrak{B}$ of f. Then $\deg(f) = 1$.*

Proof. We consider the local complete intersection curve $f^{-1}(\ell)$. As in [Sch99, p. 225], we have

$$f^{-1}(\ell) \cdot \mathcal{O}_X(1) = \frac{d}{\mu} \tag{7.6.1}$$

In particular, the curve $f^{-1}(\ell)$ contains at most $\frac{d}{\mu}$ irreducible components. If $\tilde{\ell} \to f^{-1}(\ell)$ is the normalization, this implies

$$-2\frac{d}{\mu} \leq \deg \omega_{\tilde{\ell}} \tag{7.6.2}$$

On the other hand, a standard computation, again taken from [Sch99], shows that

$$\deg \omega_{f^{-1}(\ell)} = (r-2)d - r\frac{d}{\mu}. \tag{7.6.3}$$

Recall that $\deg \omega_{\tilde{\ell}} \leq \deg \omega_{f^{-1}(\ell)}$ with equality if and only if $f^{-1}(\ell)$ is normal. Combining Equations (7.6.2) and (7.6.3), we obtain

$$-2\frac{d}{\mu} \leq \deg \omega_{\tilde{\ell}} \leq \deg \omega_{f^{-1}(\ell)} = (r-2)d - r\frac{d}{\mu}. \tag{7.6.4}$$

Case 1: $r = 1$. In this case Equation (7.6.4) immediately gives $\mu = 1$. Hence $\deg f = 1$.

Case 2: $r = 2$. In this case, the left- and right hand side of (7.6.4) are equal. The equality of $-2\frac{d}{\mu} = \deg \omega_{\tilde{\ell}}$ implies that $\tilde{\ell}$ contains precisely $\frac{d}{\mu}$ components, each isomorphic to $\mathbb{P}_1$. The equality in $\deg \omega_{\tilde{\ell}} \leq \deg \omega_{f^{-1}(\ell)}$ says that $f^{-1}(\ell)$ is smooth and (7.6.1) asserts that the components of $f^{-1}(\ell)$ are disjoint lines. To conclude, apply [HM01, Lemma 4.2]. $\square$

Remark 7.7. If $r = 2$ and there exists a line ℓ with trivial normal bundle, then [HM03] proves that $\deg f = 1$. However without assumption on the normal bundle it might a priori happen that the deformations of a line ℓ are all contained in a divisor.

7.C. **Positivity of $\mathcal{E}_f$.** The next criteria for X being isomorphic to the projective space will generally rely on positivity properties of the bundle $\mathcal{E}_f$, or of its restrictions to rational curves.

The following Theorem 7.8 and its immediate Corollary 7.9 are general recipies for answering Problem 7.1 positively. Recall that a rational curve $C \subset X$ with normalization $\eta : \mathbb{P}_1 \to C$ is called *standard*, if $\eta^*(T_X) \cong \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus b}$ for some $a, b \geq 0$.

Theorem 7.8. *Under the Assumptions 7.2, $X \cong \mathbb{P}_n$ if and only if there exists a number k such that both of the following two conditions hold.*

(7.8.1) *There exists a covering family $(C_t)_{t \in T}$ of curves with T an irreducible component of the Chow scheme, such that C_t is a standard rational curve for general t and such that $\mathcal{E}_{f_k}|_{C_t}$ is ample for all t, and*

(7.8.2) $h^0\big(X,\, f_k^*(T_X)\big) > h^0\big(X,\, T_X\big)$.

Proof. First suppose that $X \cong \mathbb{P}_n$, and let $k \geq 1$ be any number. Condition (7.8.1) follows from [Laz04, Thm. 6.3.55]. Condition (7.8.2) results from lifting back the Euler sequence via f_k.

Now suppose conversely that a number k is given such that (7.8.1) and (7.8.2) hold. Consider the splitting

$$\begin{aligned} H^0\big(X,\, (f_k)^*(T_X)\big) &\cong H^0(X,\, (f_k)_*(f_k)^*(T_X)\big) \\ &\cong H^0\big(X,\, T_X \otimes (f_k)_*(\mathcal{O}_X)\big) \\ &\cong H^0\big(X,\, T_X \oplus (T_X \otimes \mathcal{E}^*_{f_k})\big) \\ &\cong H^0\big(X,\, T_X\big) \oplus H^0(X,\, T_X \otimes \mathcal{E}^*_{f_k}\big). \end{aligned}$$

By property (7.8.2), $\mathrm{Hom}_X\big(\mathcal{E}_{f_k},\, T_X\big) \neq 0$, and we obtain a non-trivial map $\mathcal{E}_{f_k} \to T_X$. Theorem A.2 then implies that $X \cong \mathbb{P}_n$. □

The following corollary is a slightly less technical reformulation of Theorem 7.8.

Corollary 7.9. *Under the Assumptions 7.2, $X \cong \mathbb{P}_n$ if and only if there exists a number k such that both of the following two conditions hold.*

(7.9.1) *The vector bundle $\mathcal{E}_{f_k}$ is ample, and*

(7.9.2) $h^0\big(X,\, f_k^*(T_X)\big) > h^0\big(X,\, T_X\big)$.

Proof. If (7.9.1) and (7.9.2) hold, Theorem 7.8 applies. If $X \cong \mathbb{P}_n$, (7.9.2) follows from Theorem 7.8 and (7.9.1) from [Laz04, Thm. 6.3.55]. □

Condition (7.9.1) in Theorem 7.8 has a the following useful reformulation.

Lemma 7.10. *Let $C \subset X$ be a rational curve not in the branch locus $\mathfrak{B}$. The vector bundle $\mathcal{E}_f|_C$ is ample if and only if $f^{-1}(C)$ is connected.*

Proof. Let $\tilde{C}$ be the normalization with its canonical morphism $\eta : \tilde{C} \to X$, and consider the fibered product $\tilde{X} := X \times_X \tilde{C}$ with its base change diagram.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \tilde{C} & \xrightarrow[\eta]{} & X. \end{array}$$

Then $\mathcal{E}_{\tilde{f}} = \eta^*(\mathcal{E}_f)$. The fibered product $\tilde{X}$ is connected if and only if $f^{-1}(C)$ is. Since η is finite, $\mathcal{E}_{\tilde{f}}$ is ample if and only if $\mathcal{E}_f|_C$ is.

If $\tilde{X}$ is connected, [PS00, Thm. 1.3] asserts that $\mathcal{E}_{\tilde{f}}$ is ample. If $\tilde{X}$ is *not* connected, the vector space of locally constant function on $\tilde{X}$ will be at least two-dimensional. Accordingly, $\mathcal{E}_{\tilde{f}}^* = \tilde{f}_*(\mathcal{O}_{\tilde{X}})/\mathcal{O}_{\tilde{C}}$ has at least one global section, and its dual cannot be ample. □

Combining Theorem 7.8, Theorem 6.7 and Lemma 7.10, we obtain the following.

Corollary 7.11. *Let X be Fano with $\rho(X) = 1$. Let $f : X \to X$ be an endomorphism with $\deg f > 1$. Then $X \simeq \mathbb{P}_n$ if the following two conditions hold.*

(7.11.1) *f is Galois or there exists a standard rational curve $C \not\subset \mathfrak{B}$ such that $f^{-1}(C_t)$ is connected for all deformations C_t of C and no component of any C_t is contained in $\mathfrak{B}$.*

(7.11.2) $h^0\big(X,\ f^*(T_X)\big) > h^0\big(X,\ T_X\big)$.

□

7.D. **Manifolds with many vector fields.** We will now consider Condition (7.9.2) in Theorem 7.8, which means that there are infinitesimal deformations of f_k which do not come from automorphisms of X. We describe vector fields and group actions on manifolds for which $h^0(X, T_X) = h^0(X,\ f^*(T_X))$, and give criteria for Condition (7.9.2) to hold.

Proposition 7.12. *If $h^0(X, T_X) = h^0(X, f^*(T_X))$, then there exists a surjective morphism of Lie groups $\eta : \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(X)$ such that f is equivariant with respect to the natural action $\iota : \mathrm{Aut}^0(X) \times X \to X$ upstairs and the action $\iota \circ (\eta \times id)$ downstairs. In particular, the action of $\mathrm{Aut}^0(X)$ stabilizes both the ramification and the branch loci.*

Proof. Given a vector field $\vec{v} \in H^0(X, T_X)$, the assumption $h^0(X, T_X) = h^0(X, f^*(T_X))$ implies that there is a (unique) vector field $\alpha(\vec{v})$ such that

$$df(\vec{v}) = f^*(\alpha(\vec{v})), \tag{7.12.1}$$

where $df : T_X \to f^*(T_X)$ is the differential of f. Equation 7.12.1 implies that the action (upstairs) of the 1-parameter group associated with $\vec{v}$ is fiber preserving. Since $\vec{v}$ is arbitrary, the set of fiber preserving automorphisms is open in $\mathrm{Aut}^0(X)$. Since it is also closed, all automorphisms in $\mathrm{Aut}^0(X)$ are fiber preserving. The existence of η then follows, e.g., from [HO84, Prop. 1 on p. 14]. The surjectivity is immediate. □

Remark 7.12.2. The linear isomorphism $\alpha : H^0(X, T_X) \to H^0(X, T_X)$ is in fact the Lie-algebra morphism associated with the group morphism η.

7.D.1. *Zero loci of vector fields.* We will show that the assumption $h^0(X, T_X) = h^0(X,\ f^*(T_X))$ guarantees that the zero locus of any vector field X maps to the branch locus $\mathfrak{B} \subset X$ downstairs —at least for a sufficiently high iterate of f. Let $\mathfrak{B}_k$ denote the branch locus of $f_k : X \to X$. The zero-locus of a vector field $\vec{v} \in H^0(X,\ T_X)$ will be denoted $Z(\vec{v})$.

Theorem 7.13. *Under the Assumptions 7.2, suppose that $h^0\big(X,\ T_X\big) = h^0\big(X,\ f^*(T_X)\big)$ and let $\vec{v} \in H^0\big(X,\ T_X\big) \setminus \{0\}$ be a non-zero vector field. Then there exists a number k such that $f_k\big(Z(\vec{v})\big) \subset \mathfrak{B}_k$.*

Proof. We argue by contradiction and assume that $f_k\big(Z(\vec{v})\big) \not\subset \mathfrak{B}_k$ for all k. For any number k, choose an irreducible component $Z_k \subset Z(\vec{v})$ such that

(7.13.1) $Z'_k := f_k(Z_k) \not\subset \mathfrak{B}_k$, and

(7.13.2) $\dim Z_k$ is maximal among all irreducible components that satisfy (7.13.1).

Let $df_k : T_X \to f_k^*(T_X)$ be the differential of f_k. Note that if $z \in Z_k$ is a general point, then d is injective everywhere along the fiber $f_k^{-1} f_k(z)$. In particular, df_k is generically injective along all components of the subvariety $Z''_k := f_k^{-1}(Z'_k)$, which is reduced, possibly reducible but of pure dimension.

A repeated application of the assumption that $h^0\big(X, T_X\big) = h^0\big(X, f^*(T_X)\big)$ shows the existence of a vector field $\vec{w}_k \in H^0\big(X, T_X\big)$ such that $df_k(\vec{v}) = f_k^*(\vec{w}_k)$. The vector field $\vec{w}_k$ will then vanish along Z'_k. Since df_k is generically injective along all components of Z''_k, the vector field $\vec{v}$ will vanish along those components, i.e. $Z''_k \subset Z(\vec{v})$.

A repeated application of Formula 7.5.1 yields that

$$\deg Z''_k = \deg f_k^*(Z'_k) = \mu^{k\cdot(\dim X - \dim Z)} \cdot \deg Z'_k \geq \mu^{k\cdot(\dim X - \dim Z)}.$$

In particular, if $K := \limsup(\dim Z_k)$, then we obtain a subsequence $Z''_{k_i} \subset Z''_k$ such that

- the Z''_{k_i} are reduced, possibly reducible subvarieties of Z of pure dimension K, and
- the degree of the Z''_{k_i} is unbounded: $\lim(\deg Z''_k) = \infty$.

This is clearly impossible. □

For special vector fields, it is not necessary to map the zero locus down via f_k. Thus, a better statement holds.

Theorem 7.14. *Under the Assumptions 7.2, suppose that* $h^0\big(X, T_X\big) = h^0\big(X, f^*(T_X)\big) \neq 0$. *Then there exists a number* k *and a vector field* $\vec{v}_0$ *such that* $Z(\vec{v}_0) \subset \mathfrak{B}_k$.

Proof. Again, let α be the linear isomorphism discussed in Remark 7.12.2. Let $\vec{v}_0$ be an eigenvector of α and observe that $Z(\vec{v}_0) = Z(\alpha(\vec{v}_0)) =: Z$.

By Theorem 7.13, the proof is finished if we show that $Z = f(Z)$, thus $Z = f_k(Z)$ for any k. For that, it suffices to note that the equality $d(\vec{v}_0) = f^*\alpha(\vec{v}_0)$ implies that f is equivariant with respect to the flow of the vector fields $\vec{v}_0$ (upstairs) and $\alpha(\vec{v}_0)$ (downstairs). □

7.D.2. *Almost homogeneous manifolds.* For the following recall that a manifold X is *almost homogeneous* if the automorphism group acts with an open orbit. Let $E \subset X$ denote the complement of this open orbit; we call E the exceptional locus of the almost homogeneous manifold. Equivalently, E is the minimal set such that $T_X|_{X\setminus E}$ is generated by global holomorphic vector fields on X. If X is homogeneous, the set E is empty.

In this setup, Proposition 7.12 has two immediate corollaries.

Corollary 7.15. *Under the Assumptions 7.2, let X be an almost homogeneous projective manifold. Let $E_d \subset E$ be the (reduced, possibly empty) codimension-1 part of E and write $E_d = \mathcal{O}_X(a)$. Let r be the index of X and μ the number introduced in Notation 7.3. If $\mu(r-a) \geq r$, then $h^0\big(X, f^*(T_X)\big) > h^0\big(X, T_X\big)$.*

Proof. We argue by contradiction and assume $h^0\big(X, f^*(T_X)\big) = h^0\big(X, T_X\big)$. Recall from Equation 7.4.1 on page 19 that $c_1(\mathfrak{R}) = (\mu - 1)r$. By Proposition 7.12, we have an inclusion of reduced divisors $\mathfrak{B} \subseteq E_d \subset X$. Since $\mathfrak{R}$ is a strict subdivisor of $f^*(\mathfrak{B})$, we obtain the following.

$$(\mu - 1)r = c_1(\mathfrak{R}) < c_1(f^*(\mathfrak{B})) \leq c_1(f^*(E_d)) = \mu \cdot a \quad \Leftrightarrow \quad \mu(r-a) \geq r$$

□

Corollary 7.16. *In the setup of Corollary 7.15, if $r > a$, then $h^0\big(X, f_k^*(T_X)\big) > h^0\big(X, T_X\big)$ for $k \gg 0$.*

Proof. Choose k large enough so that $\mu^k(r-a) \geq r$ and apply the argumentation of Corollary 7.15 to the morphism f_k. □

Remark 7.16.1. Note that $r > a$ holds automatically if $h^0(X, T_X) > \dim X$, for the following reason. If $h^0(X, T_X) > n$, then E_d is contained in the intersection of two distinct anticanonical divisors, given by wedge products of the form $s_0 \wedge \ldots \wedge s_{n-1} = 0$ or $s_1 \wedge \ldots \wedge s_n = 0$. Thus $a < r$.

Notice also as a special case of 7.13 that if $\mathfrak{R} \not\subset E$, then $h^0(f^*(T_X)) > h^0(T_X)$.

7.E. **Manifolds that satisfy a Cartan-Fubini condition.** Let X be a Fano manifold and $T \subset \mathrm{RatCurves}^n(X)$ be a dominating family of rational curves of minimal degrees. If $x \in X$ is a general point, we can look at the set of distinguished tangent directions,

$$\mathcal{C}_x := \big\{ \vec{v} \in \mathbb{P}(T_X^*|_x) \,|\, \exists \ell \in T \text{ such that } x \in \ell \text{ and } \vec{v} \in \mathbb{P}(T_\ell|_x) \big\}.$$

We refer to [KS06, Hwa01, Keb02b] for details.

Hwang and Mok have shown that for many Fano manifolds of interest, the set of distinguished tangent directions in an analytic neighborhood of the general point gives very strong local invariants that determine the manifold globally. More precisely, they have shown that many Fano manifolds satisfy the following Cartan-Fubini property.

Definition 7.17 (see [HM01])**.** *A X Fano manifold with $\rho(X) = 1$ is said to satisfy the* Cartan-Fubini property*, (CF) for short, if there exists a dominating family $T \subset \mathrm{RatCurves}^n(X)$ of rational curves of minimal degrees such that the following holds.*

If X' is any other Fano manifold with $\rho(X') = 1$, $S \subset \mathrm{RatCurves}^n(X')$ any dominating family of rational curves of minimal degrees and $\varphi : U \to U'$ a biholomorphic map between analytic open sets that respects the varieties of minimal rational tangents associated with T and S, then φ extends to a biholomorphic map $\Phi : X \to X'$.

We refer to [HM01] for details and for examples of Fano manifolds that satisfy (CF). It is conjectured that almost all Fano manifolds will have that property. We mention two examples that will be of interest to us.

Proposition 7.18. *Let X be a projective Fano manifolds with $b_2(X) = 1$, not isomorphic to the projective space. If one of the following holds, then the manifold X satisfies (CF).*

(7.18.1) $\dim X \geq 3$*, and X is prime of index* $> \frac{\dim X + 1}{2}$, [Hwa01, 1.2, 1.5 and 2.5]

(7.18.2) X *is rational homogeneous,* [HM01, (2) on p. 566] □

Theorem 7.19. *Under the Assumptions 7.2, suppose that X satisfies (CF) and that X is almost homogeneous. Then $h^0(X, T_X) = n$ and both the branch– and the ramification divisor of f are contained in the exceptional locus $E \subset X$.*

Proof. Since X satisfies (CF), [HM01, Cor. 1.5] asserts that $\dim_f \mathrm{Hom}(X, X) = \dim \mathrm{Aut}^0(X)$. Corollary 7.16 and Remark 7.16.1 then show that $h^0(X, T_X) \leq n$. But since X is almost homogeneous, we have equality. Proposition 7.12 tells us where the branch and ramification loci are. □

Since rational homogeneous manifolds X always satisfy $h^0(X, T_X) > \dim X$, we obtain the following result of Paranjapé and Srinivas as an immediate corollary.

Corollary 7.20 ([PS89, HM99])**.** *Under the Assumptions 7.2, suppose that X is rational-homogeneous. Then X is isomorphic to the projective space.* □

Theorem 7.21. *Let X be a del Pezzo manifold of degree 5 of dimension $n \geq 3$. Then X does not admit an endomorphism of degree $\deg f > 1$.*

Proof. If $\dim X = 3$, then the claim is shown in [ARVdV99], [Sch99], and in [HM03].

If $\dim X > 3$, we claim that X is almost homogeneous with $h^0\big(X,\, T_X\big) > \dim X$. Proposition 7.18(7.18.1) and Theorem 7.19 then immediately show Theorem 7.21. To prove the claim, recall from [IP99, Thm. 3.3.1] that $\dim X \leq 6$. Moreover, if $\dim X = 6$, then X is the Grassmannian $G(2,5)$ and we are done by the preceding corollary. In the remaining cases $n = 4, 5$, recall from [Fuj81, Sects. 7.8–13] that there exists a diagram of birational morphisms

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\phi"] & \\ X & & \mathbb{P}_n \end{tikzcd}$$

where ϕ is the blow-up of a subvariety F which is entirely contained in a linear hypersurface $H \subset \mathbb{P}_n$. Notice that the vector fields in $\mathbb{P}_n$ fixing H pointwise span $T_{\mathbb{P}_n}$ outside H, simply because $T_{\mathbb{P}_n}(-1)$ is spanned. Since F is contained in H, these vector fields extend to $\tilde{X}$ and span $T_{\tilde{X}}$ outside $\phi^{-1}(H)$. Moreover

$$h^0(\tilde{X},\, T_{\tilde{X}}) \geq h^0(\mathbb{P}_n,\, T_{\mathbb{P}_n}(-1)) = n+1.$$

Since σ has connected fibers, the vector fields on $\tilde{X}$ descent to X, so that X is almost homogeneous with $h^0(X,\, T_X) > n$. This shows the claim and ends the proof of Theorem 7.21. $\square$

APPENDIX A. MANIFOLDS WHOSE TANGENT BUNDLES CONTAIN AMPLE SUBSHEAVES

The main result of this section is a slight generalization of a result of Andreatta and Wisniewski [AW01] that characterizes the projective space as the only Fano manifold X with $b_2(X) = 1$ whose tangent bundle contains an ample, locally free subsheaf.

A.A. **Setup and statement of result.** Throughout this section, we consider a setup where X is a Fano manifold and $T' \subset \mathrm{RatCurves}^n(X)$ a dominating family of rational curves of minimal degrees —again we refer to [KS06, Hwa01, Keb02b] for details about this notion.

Notation A.1. There exists a natural morphism $\iota : T' \to \mathrm{Chow}(X)$. Let $T := \overline{\iota(T')} \subset \mathrm{Chow}(X)$ be the closure of its image. If $t \in T$ is any point, let $C_t \subset X$ be the reduction of the associated curve. The curve C_t will then be rational, reduced, but not necessarily irreducible.

The following is the main result of this section. Its proof is given in Section A.B below.

Theorem A.2. *Let X be a Fano manifold with with $\rho(X) = 1$ and $\mathcal{F} \subset T_X$ be a subsheaf of rank r. Assume that there exists a dominating family of rational curves of minimal degrees, $T' \subset \mathrm{RatCurves}^n(X)$, such that for any point $t \in T \subset \mathrm{Chow}(X)$, the restriction $\mathcal{F}|_{C_t}$ is ample.*

Then $X \simeq \mathbb{P}_n$, and either $\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus r}$ or $\mathcal{F} \simeq T_{\mathbb{P}_n}$.

Remark A.2.1. Recall that a coherent sheaf $\mathcal{F}$ is called ample if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is ample. Details concerning this notion are discussed, e.g., in [Anc82].

Corollary A.3. *Let X be a projective manifold with $\rho(X) = 1$ and $\mathcal{F} \subset T_X$ an ample subsheaf. Then $X \simeq \mathbb{P}_n$.* $\square$

A.B. **Proof of Theorem A.2.** In the remainder of the present section, we will prove Theorem A.2. For the reader's convenience, we subdivide the proof into a number of fairly independent steps.

Step 0: Setup of notation. If $t \in T$ is any point, the associated curve C_t is a union of irreducible, rational curves. Let $f_t : \coprod \mathbb{P}_1 \to X$ be the normalization morphism. We will consider the determinant $\det(\mathcal{F}) := \wedge^{[r]}\mathcal{F} \subset \wedge^r T_X$. Finally, set $n := \dim X$.

Remark A.4. The sheaf $\mathcal{F}$ is a subsheaf of a torsion-free sheaf, and therefore torsion free itself. In particular, if $\operatorname{Sing}(\mathcal{F})$ is the singular locus of $\mathcal{F}$, i.e., the locus where $\mathcal{F}$ is not locally free, then $\operatorname{codim}_X \operatorname{Sing}(\mathcal{F}) \geq 2$.

To prove Theorem A.2, we will need to use the following description of curves associated with general points of T, [Hwa01, Thm. 1.2].

Fact A.5. A general point $t \in T$ corresponds to an irreducible rational curve C with normalization $f_t : \mathbb{P}_1 \to C$ such that

$$f^*(T_X) = \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(1)^p \oplus \mathcal{O}_{\mathbb{P}_1}^{n-p-1}. \tag{A.5.1}$$

A curve for which (A.5.1) holds, is called "standard".

Step 1: The rank and the Chern class of $\mathcal{F}$.

Lemma A.6. *Let $t \in T$ be a general point. Then either one of the following holds true.*

(A.6.1) $r = n$ *and* $c_1(\mathcal{F}) \cdot C_t = n+1$, *or*
(A.6.2) $c_1(\mathcal{F}) \cdot C_t = r$.

Proof. We argue by contradiction and assume that

$$c_1(\mathcal{F}) \cdot C_t \neq r \quad \text{and} \quad r < n. \tag{A.6.3}$$

Now consider a general point $t \in T$ and the associated irreducible, reduced curve C_t. Since $\operatorname{codim}_X \operatorname{Sing}(\mathcal{F}) \geq 2$, recall from [Hwa01, lem. 2.1] that the curve C_t does not meet the singular locus of $\mathcal{F}$, so that $f_t^*(\mathcal{F})$ is a locally free sheaf on $\mathbb{P}_1$. The ampleness assumption and Equation A.5.1 then imply that

$$f_t^*(\mathcal{F}) = \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{r-1} \quad \text{or} \quad \mathcal{O}_{\mathbb{P}_1}(1)^r.$$

In particular, we have that

$$c_1(\mathcal{F}) \cdot C_t = c_1\big(f_t^*(\mathcal{F})\big) \geq r.$$

Equation (A.6.3) then implies that $c_1(\mathcal{F}) \cdot C_t > r$. The slope μ with respect to the class of C_t therefore fulfills the inequality

$$\mu(\mathcal{F}) \geq \frac{r+1}{r} > \frac{n+1}{n} \geq \mu(T_X).$$

In particular, T_X is not semistable and $\mathcal{F}$ is destabilizing. From [Hwa98, Prop. 1 and Prop. 3], the maximal destabilizing subsheaf $\mathcal{H}$ of T_X must satisfy $\mu(\mathcal{H}) \leq 1$, contradicting $\mu(\mathcal{F}) > 1$. Hence $\mathcal{F}$ has rank n and $c_1(\mathcal{F}) \cdot C_t = n+1$. $\square$

Step 2: Proof in case (A.6.1). The following Proposition ends the proof of Theorem A.2 in case (A.6.1).

Proposition A.7. *Let $t \in T$ be a general point. If $r = n$ and $c_1(\mathcal{F}) \cdot C_t = n + 1$, then $X \cong \mathbb{P}_n$ and $\mathcal{F} = T_X$.*

Proof. The injection $\mathcal{F} \to T_X$ immediately gives an injection $\det(\mathcal{F}) \to \det(T_X)$. In particular, we have that $-K_X.C_t \geq n+1$ for all $t \in T$. In this setup, [Keb02a] gives that $X \cong \mathbb{P}_n$, that the family T' is proper and that the associated curves C_t are lines[1].

It remains to show that $\mathcal{F} = T_{\mathbb{P}_n}$. We argue by contradiction and assume that the inclusion sequence has non-zero cokernel Q,

(A.7.1) $$0 \longrightarrow \mathcal{F} \longrightarrow T_{\mathbb{P}_n} \xrightarrow{\beta} Q \longrightarrow 0.$$

The equality of Chern classes implies that the support S of Q has codimension at least 2.

To start, we claim that for any point $x \in S$, $\operatorname{rank} Q_x = 1$ as an $\mathcal{O}_{X,x}-$module. Indeed, take a general line ℓ passing through x and restrict sequence (A.7.1) to ℓ:

$$\mathcal{F}|_\ell \longrightarrow T_{\mathbb{P}_n}|_\ell \xrightarrow{\beta|_\ell} Q|_\ell \longrightarrow 0.$$

If Q_x had a rank larger than 1, then the splitting $T_{\mathbb{P}_n}|_\ell \cong \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{n-1}$ immediately implies that the sheaf $\mathcal{F}_\ell$ cannot be ample. This shows the claim that $\operatorname{rank} Q_x = 1$.

Now choose a point $x \in S$ and consider the surjective quotient map at x, $\beta|_x : T_{\mathbb{P}_n}|_x \to Q|_x$. Here $Q|_x$ is a $1-$dimensional vector space so that its kernel $\ker(\beta|_x)$ has dimension $n-1$. The set of lines

$$M := \{\ell \subset \mathbb{P}_n \text{ a line } \mid x \in \ell \text{ and } T_\ell|_x \subset \ker(\beta|_x)\}$$

then has dimension $n-2$, and the associated lines cover a divisor in $\mathbb{P}_n$. In particular, if $\ell \in M$ is a general element, then ℓ is not contained in S. We fix a general $\ell \in M$ for the sequel.

Let Q' denote the trivial extension of Q_x to X, let $\gamma : T_{\mathbb{P}_n} \to Q'$ be the quotient map and let $\mathcal{G} = \ker(\gamma)$ be its kernel. Since $\ell \not\subset S$, we know that $\mathcal{F}|_\ell/\mathrm{tor}$ is a subsheaf of $\mathcal{G}|_\ell/\mathrm{tor}$. By choice of ℓ, the composed map $T_\ell \to T_X|_\ell \to Q'$ vanishes, hence $T_\ell \cong \mathcal{O}_{\mathbb{P}_1}(2)$ is a subsheaf of $\mathcal{G}|_\ell/\mathrm{tor}$. But then $\mathcal{G}|_\ell/\mathrm{tor}$ has a factor $\mathcal{O}_{\mathbb{P}_1}(2)$ and since $\mathcal{G}|_\ell/\mathrm{tor}$ is a proper subsheaf of $T_{\mathbb{P}_n}|_\ell \cong \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{n-1}$ of rank n, it cannot be ample. It follows that $\mathcal{F}|_\ell$ cannot be ample, a contradiction. Thus $Q = 0$ and $\mathcal{F} = T_{\mathbb{P}_n}$. □

Step 3: Proof in case (A.6.2). We compare the sheaf $\mathcal{F}_1 := \wedge^r \mathcal{F}/_{\mathrm{tor}}$ with the locally free sheaf $\det \mathcal{F} = \wedge^{[r]} \mathcal{F}$. Since $\mathcal{F}_1$ is torsion free and has rank one, recall from [OSS80, Lem. 1.1.8 on p. 147] there exists a subscheme $Z \subset X$ such that

(A.7.2) $$\mathcal{F}_1 \cong \mathcal{I}_Z \otimes \det \mathcal{F}.$$

Lemma A.8. *Let $t \in T$ be any point and consider the associated cycle C_t. If $C_t \not\subset Z$, then the cycle associated with t is irreducible, reduced and $C_t \cap Z = \emptyset$.*

In particular, if $t \in T$ is any point, then either $C_t \subset Z$, or $C_t \cap Z = \emptyset$.

[1]The statement of [Keb02a, Thm. 1.1] assumes that the inequality $-K_X.\ell \geq n+1$ holds for any curve $\ell \subset X$. Observe, however, that the proof of [Keb02a, Thm. 1.1] only uses curves ℓ coming from a given dominating family of rational curves of minimal degrees.

Proof. To show that the cycle associated with t is irreducible and reduced, we argue by contradiction and assume that there exists a component $C'_t \subset C_t$ such that $C'_t \not\subset Z$ and such that $c_1(\mathcal{F}).C'_t < r$. If $\eta : \mathbb{P}_1 \to X$ is the normalization of C'_t, then we have a natural, non-trivial inclusion

$$\mathcal{F}' := \left(\eta^*(\mathcal{F}_1) \big/ \mathrm{tor}\right) \to \eta^*(\det \mathcal{F}). \tag{A.8.1}$$

More precisely, the isomorphism (A.7.2) then shows that

$$\mathcal{F}' \cong \mathcal{I}_{\eta^{-1}(Z)} \otimes \eta^*(\det \mathcal{F}). \tag{A.8.2}$$

The morphism (A.8.1) shows that the locally free sheaf $\mathcal{F}'$ on $\mathbb{P}_1$ has rank r and degree $\deg \mathcal{F}' \leq \deg \eta^*(\det \mathcal{F}) < r$, so $\mathcal{F}'$ cannot be ample. On the other hand, $\mathcal{F}'$ is a quotient of a wedge power of the pull-back of an ample sheaf under a finite morphism, hence ample. This contradiction shows $C'_t = C_t$.

We can now assume that C_t is irreducible and that $C_t.c_1(\mathcal{F}) = r$. Again, let $\eta : \mathbb{P}_1 \to X$ be the normalization, and consider the morphism (A.8.1). Again, $\mathcal{F}'$ is ample, but this time $\deg \mathcal{F}' \leq \deg \eta^*(\det \mathcal{F}) = r$. Obviously, if $\mathcal{F}'$ is ample, then $\deg \mathcal{F}' \geq r$. Isomorphism (A.8.2) asserts that this is the case if and only if $\eta^{-1}(Z) = \emptyset$, as claimed. □

Corollary A.9. *The subscheme $Z \subset X$ is empty. In particular, the family T is non-split, i.e., if $t \in T$ is any point, then the associated curve C_t is irreducible and satisfies $C_t.c_1(\mathcal{F}) = r$.*

Proof. By Lemma A.8, both Z and $X \setminus Z$ are unions of curves of the form C_t. Fix a general point $x \in X \setminus Z$ and consider the compact variety $Z^{[1]}$ filled up by all curves C_t through x. Then consider the compact variety $Z^{[2]}$ filled up by all curves C_t meeting $Z^{[1]}$ and so on. By Lemma A.8, all $Z^{[k]}$ are contained in $X \setminus Z$. Now the sequence $(Z^{[k]})_{k \in \mathbb{N}}$ must stabilize at some k_0.

Assume that $Z \neq \emptyset$. Then $Z^{[k_0]}$ is a proper subvariety of X. In that case, the family (C_t) is not connecting, and we obtain an almost holomorphic map $X \dashrightarrow W$ to a normal variety W of positive dimension. This contradicts $\rho(X) = 1$ and shows the claim. □

Corollary A.10. *The double dual $\mathcal{F}^{**}$ is locally free.*

Proof. Following [GR70, Satz 1.1, Rossi's Theorem], there exists a sequence of blowups $\pi : \widehat{X} \to X$ such that $\pi^*(\mathcal{F})/\mathrm{tor}$ is locally free.

$$\begin{aligned}
\pi^* \det(\mathcal{F}) &\cong \pi^*(\wedge^r \mathcal{F}/\mathrm{tor}) \\
&\cong \pi^*(\wedge^r \mathcal{F})/\mathrm{tor} && \text{by [GR70, Satz 1.3]} \\
&\cong \wedge^r(\pi^* \mathcal{F})/\mathrm{tor} \\
&\cong \wedge^r(\pi^* \mathcal{F}/\mathrm{tor})/\mathrm{tor} \\
&\cong \wedge^r(\pi^* \mathcal{F}/\mathrm{tor}) \cong \det(\pi^* \mathcal{F}/\mathrm{tor}) && \text{since } \pi^*\mathcal{F}/\mathrm{tor} \text{ is locally free}
\end{aligned}$$

In particular, if $\widehat{X}_x$ is any π-fiber, then the restriction $\det(\pi^* \mathcal{F}/\mathrm{tor})|_{\hat{X}_x} \cong \mathcal{O}_{\hat{X}_x}$ is trivial. Since $(\pi^* \mathcal{F}/\mathrm{tor})|_{\hat{X}_x} \subset \pi^*(T_X)|_{\hat{X}_x} \cong \mathcal{O}_{\hat{X}_x}^{\oplus \dim X}$, it follows that $(\pi^* \mathcal{F}/\mathrm{tor})|_{\hat{X}_x} \cong \mathcal{O}_{\hat{X}_x}^{\oplus r}$.

Consequently, there exists a vector bundle $\mathcal{F}'$ on X such that $\pi^* \mathcal{F}/\mathrm{tor} \cong \pi^*(\mathcal{F}')$. The sheaves $\mathcal{F}^{**}$ and $\mathcal{F}'$ are isomorphic outside the singular locus of $\mathcal{F}$, which of codimension at least two, hence $\mathcal{F}^{**} = \mathcal{F}'$ is locally free. □

Corollary A.11. *The sheaf $\mathcal{F}$ is locally free.*

Proof. Consider the natural exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{**} \to Q \to 0, \tag{A.11.1}$$

where Q is a torsion sheaf. We need to show that $Q = 0$.

Let $t \in T$ be any point and C_t the associated curve which, by Corollary A.9, is irreducible. We claim that either $C_t \subset \mathrm{Supp}(Q)$, or that C_t is disjoint from $\mathrm{Supp}(Q)$. Indeed, if C_t would meet $\mathrm{Supp}(Q)$ in a finite non-empty set, and if $\eta : \mathbb{P}_1 \to X$ is the normalization morphism, Sequence (A.11.1) pulls back to

$$\eta^*(\mathcal{F})/\mathrm{tor} \to \underbrace{\eta^*(\mathcal{F}^{**})}_{\cong \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}} \to \underbrace{\eta^*(Q)}_{\not\cong 0} \to 0.$$

In particular, $\eta^*(\mathcal{F})/\mathrm{tor}$ is a locally free strict subsheaf of $\mathcal{O}_{\mathbb{P}_1}(1)^{\oplus r}$ and cannot be ample. This shows the claim.

As in the proof of Corollary A.9, consider the varieties $Z^{[k]}$ covered by curves that, observe that the $Z^{[k]}$ do not intersect $\mathrm{Supp}(Q)$ and conclude that $\mathrm{Supp}(Q)$ is empty. □

We have shown that $\mathcal{F}$ is locally free. To finish the proof of Theorem A.2, apply [AW01].

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